OLG

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1 The OLG Model

2 Syllabus

This course provides a comprehensive introduction to the Overlapping Generations (OLG) model, one of the most versatile and influential frameworks in modern macroeconomics. Unlike the Ramsey model with infinitely-lived agents, the OLG framework captures the realistic life-cycle dynamics where individuals live for finite periods, making decisions about consumption, savings, and intergenerational transfers.

The OLG model is particularly powerful for analyzing issues that involve intergenerational considerations, such as social security systems, public debt sustainability, educational policies, and long-run economic development. The course combines theoretical foundations with applications to contemporary economic research, demonstrating how this framework helps us understand fundamental questions about economic growth, inequality, and policy design.

Through the study of cutting-edge research papers, we explore how economists use the OLG framework to investigate diverse phenomena from the evolution of social preferences to environmental sustainability.

2.1 Course Objectives

Upon completion of this course, students will be able to:

Theoretical Foundations: - Master the core OLG model with its key assumptions and derive competitive equilibria - Understand the fundamental differences between OLG and Ramsey models, particularly regarding dynamic efficiency - Analyze steady-state properties, stability conditions, and the possibility of multiple equilibria - Comprehend the golden rule and its welfare implications

Technical Skills: - Solve household optimization problems in a life-cycle context with perfect foresight - Characterize intertemporal equilibria and analyze their existence and uniqueness - Evaluate the welfare properties of competitive equilibria and identify potential market failures

Research Applications: - Critically analyze how contemporary researchers extend the basic OLG framework - Understand applications to cultural evolution, environmental economics, and development - Develop skills to formulate and solve original research questions using the OLG framework

2.2 Prerequisites

- Intermediate Microeconomics (consumer theory, optimization)
- Mathematical Methods for Economics (differential calculus, basic dynamic systems)
- Introduction to Macroeconomics
- Basic knowledge of econometrics is helpful but not required

2.3 Course Format

- Duration: 10 sessions (2 hours per session)

2.4 Assessment

Student evaluation consists of:

- **Final Examination (100%)**: Comprehensive exam covering theoretical concepts, model applications, and analysis of research papers discussed in class

2.5 Contact Information

- Instructor: Eric Roca (eric.roca_fernandez@uca.fr)
- Office Hours: By appointment

2.6 Bibliography

2.6.1 Core Textbooks

- Croix and Michel (2009)
- Campante et al. (2021) Freely available online

2.6.2 Research Papers

- Diamond (1965)
- Galor and Moav (2006)
- Galor and Özak (2016)
- Croix and Dottori (2008)

3 The OLG model

3.1 Introduction

Based on Croix and Michel (2009).

We have studied the Ramsey model, which predicts that the path derived from the competitive equilibrium is optimal. One of the main features of this model is that agents have an infinite horizon: they live forever and optimise considering an infinite horizon.

The overlapping generations model changes this hypothesis and focuses on the life cycle: agents make decisions regarding how much to consume and save for retirement. That is, the OLG model assumes that agents work until some age and then retire.[1] A focus of the OLG model is intergenerational redistribution, which allows us to study:

- social security,
- education policies and
- public debt.

The main departure with respect to the Ramsey model is that in OLG, agents are heterogeneous. Individuals live for two periods of time, and then die. In the first period, they are young and work. When old, they

[1] Bisin and Verdier also model how parental indoctrination affects the evolution of cultural traits.

retire and live from savings. Hence, at any point in time, *two types of agents with **different** budget constraints exist*: young and old.[2]

As we shall see, in this model the competitive path *may not be optimal.* Consequently, there may be instances in which the utility of all individuals can be increased. Therefore, the OLG model opens the door to government intervention, to reallocate consumption and savings efficiently. The OLG framework also permits the existence of bubbles and fluctuations.

A basic reference for this model is Diamond (1965)

3.2 Preliminaries

In this model, time is discrete and extends from $t = 0, 1, \ldots, \infty$. Individuals make decisions at points in time. We shall have initial conditions detailing the state of the economy at $t = 0$.

3.2.1 Individuals live for two periods

The main difference with respect to the Ramsey model is that in the OLG model, individuals live for two periods. **Note:** this means that, at every point in time, two generations are alive and overlap.

This is relevant: the economy goes on forever but individuals only operate during some periods. Hence, there will be infinite two-period-lived generations. In particular, at $t = 0$, we have a young generation born at $t = 0$ and an old generation born at $t = -1$. The old generation dies at the end of $t = 0$; at $t = 1$, the generation born at $t = 0$ is old and a new young generation is born. Hence, we can represent the generations diagrammatically —in brackets I have denoted the year in which each generation was born.

[2]The assumption $\ln(R^1) > 2\ln(R^0)$ is important to establish that $\hat{\beta} \in (0, 1)$.

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
Old (t=-1)	Die				
Young(t=0)	Old (t=0)	Die			
Young(t=1)	Old (t=1)	Die			
	Young(t=2)	Old (t=2)	Die		
		Young(t=3)	Old (t=3)	Die	
			Young(t=4)	Old (t=4)	
				Young(t=5)	

Let N_t denote the size of the young cohort born at t. To simplify the model, we assume that cohort size grows at the constant rate $n > -1$. **Note:** fertility and cohort growth are exogenous. More complex set-ups include endogenous fertility. Thus,

$$N_t = (1 + n)N_{t-1}.$$

The size of the young cohort is therefore

$$N_t = N_0(1 + n)^t.$$

Because the old cohort at t has size N_{t-1}, the total living population is $N_t + N_{t-1}$, not N_t.

3.3 Assumptions

3.3.1 Firms

We assume that a large number of identical firms populate the economy. Firms produce a single, homogeneous good using labour and capital. The production function $F(K, L)$ has the following properties:

Assumption OLG 1: The production function satisfies the following properties:

- **OLG 1.1** $F(K, L)$ is continuous and defined on $[0, +\infty)^2$,
- **OLG 1.2** $F(K, L)$ has continuous derivatives of every required order on $(0, +\infty)^2$,
- **OLG 1.3** The production function is strictly increasing in both arguments: $F_i(K, L) > 0$,
- **OLG 1.4** The production function is concave, and its intensive form is strictly concave: $f''(k) < 0$ for $k > 0$.
- **OLG 1.5** $F(K, L)$ is homogeneous of degree one.
- **OLG 1.6** $F(K, L)$ satisfies the Inada conditions:

$$\lim_{K \to 0} F'_K(K, L) = \lim_{L \to 0} F'_L(K, L) = +\infty$$
$$\lim_{K \to +\infty} F'_K(K, L) = \lim_{L \to +\infty} F'_L(K, L) = 0.$$

Note: Constant returns to scale make strict concavity of F on the whole input space impossible: along any ray, output is linear in scale. Concavity of F and strict concavity of f provide the curvature used below.

Firms maximise real profits. Since there are many firms competing, in equilibrium they make exactly zero profits. Moreover, in equilibrium factors are paid their marginal productivity. Since the production function F is homogeneous of degree one, we can write it in *intensive* terms:

$$f(k) \equiv F\left(\frac{K}{L}, 1\right), \quad k \equiv \frac{K}{L}. \tag{3.1}$$

Because markets are competitive, capital earns the rental rate $r_t = \partial F(K_t, L_t)/\partial K_t$, or $f'(k_t)$ in intensive terms. After depreciation, the gross return on one unit saved is:

$$r_t = f'(k_t), \qquad R_t = 1 - \delta + r_t = 1 - \delta + f'(k_t). \tag{3.2}$$

The marginal product of labour is given by $\partial F(K, L)/\partial L$. In intensive terms, it is equal to:

$$w_t = f(k) - f'(k)k. \tag{3.3}$$

3.3.2 Households

Individuals live for two periods. As before, we assume perfect foresight for individuals. **Assumption OLG 2** Individuals have perfect foresight.

When **young**, they are endowed with *one unit of labour* that they *supply inelastically.* **Assumption OLG 3** Individuals supply one unit of labour inelastically when young. They receive the current wage rate w_t and allocate this income between:

- current consumption c_t,
- savings s_t that are invested in the firms.

Therefore, the budget constraint of a *young* individual in period t is:

$$w_t = c_t + s_t.$$

Once an individual reaches old age in the next period, the individual consumes the gross return on savings and then dies. Old people *do not care* about anything happening after death. Old-age consumption, d_{t+1}, is determined by the saving choice made when young.

The budget constraint for this period is:

$$s_t(1 - \delta + r_{t+1}) = d_{t+1}.$$

with $\delta \in (0, 1)$ being the capital depreciation rate.

Hence, an individual faces two budget constraints. However, we can collapse both into a unique *intertemporal budget constraint.*

3.3.2.1 The intertemporal budget constraint

In the economy, we have consumption as the numeraire. It is more convenient for us to combine the two budget constraints corresponding to young and old ages into one single constraint. Starting from

$$\begin{cases} w_t = c_t + s_t \\ d_{t+1} = s_t(1 - \delta + r_{t+1}) = s_t R_{t+1} \end{cases} \tag{3.4}$$

where $R_{t+1} \equiv 1 - \delta + r_{t+1} = 1 - \delta + f'(k_{t+1})$ represents the gross return on savings from t to $t + 1$, isolate s_t in the second equation and plug it into the first one:

$$w_t = c_t + \frac{d_{t+1}}{R_{t+1}}. \tag{3.5}$$

The intertemporal budget constraint indicates that the total present value of income (w_t, the only source of income) equals the total present value of expenditures. The present value of consumption when old d_{t+1} is discounted using the gross return R_{t+1}.

Savings will be a function of wages w and the gross return R. So will consumption at all periods of time.

3.3.2.2 Utility function

We suppose that the life-cycle utility function is additively separable:

$$U(c,d) = u(c) + \beta u(d),\ \beta \in (0,1) \tag{3.6}$$

where $\beta \in (0,1)$ is the psychological discount factor. We assume that $u(c)$ has the properties

Assumption OLG 4

- **OLG 4.1** $u'(c) > 0$,
- **OLG 4.2** $u''(c) < 0$,
- **OLG 4.3** $\lim_{c\to 0} u'(c) = +\infty$,
- **OLG 4.4** $\lim_{c\to +\infty} u'(c) = 0$.

The last assumption $\lim_{c\to 0} u'(c) = +\infty$ implies that an individual will always have a positive consumption —as long as he has enough income to finance it.

Another important implication of the choice of the utility formulation is that c and d are normal goods: the demand is *not decreasing* in wealth. It follows from additive separability and concavity.

3.3.3 The behaviour of individuals

At time t, young individuals receive their wages, consume and save while maximising the utility function.

$$\begin{aligned} &\max u(c_t) + \beta u(d_{t+1}) \\ \text{s.t.} \quad & w_t = c_t + s_t \\ & d_{t+1} = R_{t+1} s_t \\ & c_t \geq 0, d_{t+1} \geq 0. \end{aligned} \tag{3.7}$$

We have two possibilities to solve this problem:

3.3.3.1 Substitution

First, we can substitute c_t and d_{t+1} in the utility function, leading to:

$$u(w_t - s_t) + \beta u(R_{t+1} s_t).$$

This function is strictly concave with respect to s_t because of our assumptions. The solution is *the savings function*:

$$s_t = s(w_t, R_{t+1}).$$

The solution is interior as a consequence of the assumptions, and it is characterised by the first-order condition:

$$u'(w_t - s_t) = \beta R_{t+1} u'(R_{t+1} s_t). \tag{3.8}$$

3.3.3.2 Lagrangian

Instead, we can use the intertemporal budget constraint and build the Lagrangian:

$$\mathcal{L} = u(c_t) + \beta u(d_{t+1}) + \lambda_t (w_t - c_t - \frac{d_{t+1}}{R_{t+1}}).$$

The first order conditions imply that:

$$u'(c_t) = \lambda_t, \quad \beta u'(d_{t+1}) = \frac{\lambda_t}{R_{t+1}}.$$

Combining both, we obtain Equation 3.8 again:

$$u'(c_t) = \beta R_{t+1} u'(d_{t+1}).$$

3.4 Intertemporal elasticity of substitution

The intertemporal elasticity of substitution is defined as:

$$\epsilon_{d_{t+1},c_t} = \frac{\partial \left(\frac{d_{t+1}}{c_t}\right)}{\partial R_{t+1}} \frac{R_{t+1}}{\left(\frac{d_{t+1}}{c_t}\right)}. \tag{3.9}$$

From the Euler equation we have $u'(c_t)/u'(d_{t+1}) = \beta R_{t+1}$. In general, this condition does not make d_{t+1}/c_t a function of R_{t+1} alone: both consumption levels can change with the return and wealth. Consequently, the inverse-RRA formula does not follow from an unrestricted total differentiation that holds c_t fixed.

For CIES (equivalently, isoelastic or CRRA) period utility, homotheticity makes the consumption ratio depend only on the gross return. Let $d_{t+1} = x_{t+1} c_t$. Then the intertemporal elasticity of substitution becomes:

$$\epsilon_{d_{t+1},c_t} = \frac{\partial x_{t+1}}{\partial R_{t+1}} \frac{R_{t+1}}{x_{t+1}}.$$

For the CIES specification below, this elasticity is constant and equals the inverse of the constant coefficient of relative risk aversion. This mathematical equivalence is useful even though the deterministic model itself contains no risk.

3.4.1 CIES example

We illustrate the previous concept using a constant intertemporal elasticity of substitution utility function:

$$u(c) = \begin{cases} \dfrac{c^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}}, & \sigma > 0,\ \sigma \neq 1, \\ \log c, & \sigma = 1. \end{cases}$$

In this case, we have the following Euler condition:

$$u'(c_t) = \beta R_{t+1} u'(d_{t+1}) \implies \frac{d_{t+1}}{c_t} = \left(\beta R_{t+1}\right)^{\sigma}.$$

Therefore, the intertemporal elasticity of substitution is

$$\epsilon_{d_{t+1},c_t} = \frac{\partial \frac{d_{t+1}}{c_t}}{\partial R_{t+1}} \frac{R_{t+1}}{\frac{d_{t+1}}{c_t}} = \sigma(\beta R_{t+1})^{\sigma-1} \beta R_{t+1} \frac{1}{(\beta R_{t+1})^{\sigma}} = \sigma.$$

Meanwhile, the coefficient of relative risk aversion

$$RRA(c) = -\frac{u''(c)}{u'(c)} c = \frac{1}{\sigma} = \epsilon^{-1}_{d_{t+1},c_t}$$

is the inverse of the intertemporal elasticity of substitution.

3.5 The savings function

The savings function arises by solving:

$$s(w, R) = \arg\max \left[u(w - s) + \beta u(Rs)\right].$$

Taking the first derivative with respect to s and solving provides an implicit function: the savings function.

$$u'(w - s)(-1) + \beta u'(Rs)R = 0.$$

Denote this function $\phi(s, w, R)$:

$$\phi(s, w, R) = -u'(w - s) + \beta R u'(Rs) = 0.$$

Following from the assumption on the utility function, the savings function is continuous and continuously differentiable.

We are interested in determining whether savings increase or decrease with wealth and the gross return.

3.5.1 The effect of wages

We begin by analysing the effect of wages on savings:

$$\begin{aligned}
\frac{\partial s}{\partial w} &= -\frac{\partial \phi / \partial w}{\partial \phi / \partial s} \\
&= \frac{u''(w - s)}{u''(w - s) + \beta R^2 u''(Rs)} \\
&= \frac{1}{1 + \beta R^2 \frac{u''(Rs)}{u''(w-s)}} \in (0, 1).
\end{aligned}$$

We thus have that the marginal propensity to save out of income is between 0 and 1: $0 < s'_w < 1$, which reflects the fact that consumption goods are normal goods.

3.5.2 The effect of the gross return

Similarly, define the local elasticity $\sigma(d) \equiv -u'(d)/[du''(d)]$ and compute $\frac{\partial s(w,R)}{\partial R}$:

$$\frac{\partial s}{\partial R} = -\frac{\beta u'(d) + \beta R s u''(d)}{u''(c) + \beta R^2 u''(d)} = -\frac{\beta u'(d)\left[1 - \frac{1}{\sigma(d)}\right]}{u''(c) + \beta R^2 u''(d)}.$$

Hence,

$$\frac{\partial s}{\partial R} \gtreqless 0 \quad \text{if} \quad \sigma(d) \gtreqless 1.$$

Two effects interact in this derivative:

- Wealth effect: if the gross return rises, we obtain more from the same savings and hence become wealthier. We consume more of all goods —c_t and d_{t+1} are normal goods.
- Substitution effect: it is more profitable to save and consume d_{t+1}, inducing savings.

When the intertemporal elasticity of substitution is lower than 1, the substitution effect is dominated by the income effect. In that case, a rise in the rate of return has a negative effect on savings. When the intertemporal elasticity of substitution is higher than 1, households are ready to exploit the rise in the remuneration of savings by consuming relatively less today. In this case, raising the rate of return boosts savings. When the intertemporal elasticity of substitution is equal to 1 (log utility), the income effect exactly offsets the substitution effect and the gross return does not affect savings.

3.5.2.1 Example using a CIES function

Under a CIES utility, we have that:

$$s(w,R) = \frac{1}{1+\beta^{-\sigma}R^{1-\sigma}}w,$$

and

$$s'_w = \frac{1}{1+\beta^{-\sigma}R^{1-\sigma}} > 0.$$

However, $s'_R = -\frac{w\beta^{-\sigma}R^{-\sigma}}{(1+\beta^{-\sigma}R^{1-\sigma})^2}(1-\sigma) \gtreqless 0$ depending on $\sigma \lesseqgtr 1$.

Log-utility:

In the case of logarithmic utility, savings are independent of the gross return R and are linear in wages. Log-utility is a special case of the CIES function when $\sigma = 1$. In this case, the wealth and substitution effects cancel each other:

$$s(w,R) = \frac{\beta}{1+\beta}w.$$

3.5.2.2 Example using a CES function

Note: These homothetic preferences represent the same intratemporal trade-off as an appropriate monotonic transformation, so the solution looks similar to the CIES case.

$$U(c,d) = \left[\alpha c^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha)d^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}.$$

From here, substituting $c = w - s$ and $d = Rs$, and solving for the optimal savings we obtain:

$$
\begin{aligned}
&\alpha(w-s)^{-\frac{1}{\epsilon}} = (1-\alpha)R(Rs)^{-\frac{1}{\epsilon}} \\
&s = \frac{\left[\frac{\alpha}{1-\alpha}\right]^{-\epsilon} w}{R^{1-\epsilon} + \left[\frac{\alpha}{1-\alpha}\right]^{-\epsilon}} = \frac{w}{1 + \left[\frac{\alpha}{1-\alpha}\right]^{\epsilon} R^{1-\epsilon}}.
\end{aligned}
$$

Therefore,

$$
s'_R = -\frac{w}{\left(1 + \left[\frac{\alpha}{1-\alpha}\right]^{\epsilon} R^{1-\epsilon}\right)^2} \left(\frac{\alpha}{1-\alpha}\right)^{\epsilon} (1-\epsilon)R^{-\epsilon}.
$$

Then, when $\epsilon > 1$ savings increase with the interest rate: $s'_R > 0$. In that case, *the substitution effect dominates.* **Alternative interpretation:** ϵ is the elasticity of intertemporal substitutability, hence if it is large, individuals are willing to substitute present consumption for future consumption. For this isoelastic specification, $1/\epsilon$ is also the curvature coefficient commonly called relative risk aversion. Here, the economic interpretation concerns willingness to substitute consumption across dates, since the model is deterministic.

Conversely, when $\epsilon < 1$, an increase in the interest rate lowers savings: $s'_R < 0$. In that case, the wealth effect dominates. $\epsilon < 1$ also means that c and d are relatively difficult to substitute across dates, so their ratio responds little to changes in the return.

3.6 Temporary equilibrium

Before turning to the intertemporal equilibrium and the analysis of the steady state, we study the temporary equilibrium that takes place every period.

We have not discussed the firms, but they follow the same setup as in Ramsey: use capital and labour in a perfectly competitive environment.

We shall work in intensive form: $k_t \equiv \frac{K_t}{N_t}$.

1. **Labour market equilibrium:** *Only* young individuals supply labour. Moreover, they do so *inelastically*. During period t there are N_t young agents and, hence, the supply of labour is N_t. Equating this to labour demand from firms L_t gives the wage rate:

$$w_t = \omega\left(\frac{K_t}{N_t}\right) = \omega(k_t).$$

2. **Capital market:** *Only* old individuals own capital. The capital installed at t is owned by the old, so asset-market clearing requires

$$K_t = N_{t-1}s_{t-1}.$$

Competitive firms pay the rental rate $r_t = f'(k_t)$. Including the undepreciated principal, old households receive

$$N_{t-1}d_t = N_{t-1}R_t s_{t-1} = R_t K_t, \qquad R_t = 1 - \delta + f'(k_t).$$

3. **Good market:** Finding the equilibrium for this market departs from the otherwise similar Ramsey case. Remember that we have *two* types of agents: young and old. Total production is given by:

$$Y_t = F(K_t, N_t) = N_t f(k_t).$$

Available goods combine current output and undepreciated capital. They are used for the consumption of the old and young and for next period's capital. Thus, the resource constraint is

$$Y_t + (1-\delta)K_t = N_{t-1}d_t + N_t c_t + K_{t+1}.$$

Equivalently, if gross investment is $I_t \equiv K_{t+1} - (1-\delta)K_t$, then

$$Y_t = N_{t-1}d_t + N_t c_t + I_t.$$

Asset-market clearing for the savings of the young requires $K_{t+1} = N_t s_t$, so equivalently

$$Y_t + (1-\delta)K_t = N_{t-1}d_t + N_t(c_t + s_t).$$

We can verify this identity from factor payments:

$$\begin{aligned} N_t(c_t + s_t) &= N_t w_t \\ &= N_t \left[f(k_t) - k_t f'(k_t)\right] \\ &= Y_t - K_t f'(k_t). \end{aligned}$$

$$N_{t-1}d_t = R_t K_t = \left[1 - \delta + f'(k_t)\right] K_t.$$

Net investment is $\Delta K_t \equiv K_{t+1} - K_t = I_t - \delta K_t$. Savings finance ownership of the entire next-period capital stock, $K_{t+1} = N_t s_t$, not just net investment.

A temporary equilibrium is a set
$w_t, R_t, K_t, L_t, Y_t, k_t, I_t, c_t, s_t, d_t$ that satisfies:

$$
\begin{aligned}
w_t &= \omega(k_t), \\
R_t &= 1 - \delta + f'(k_t), \\
L_t &= N_t, \\
Y_t &= N_t f(k_t), \\
Y_t + (1-\delta)K_t &= N_{t-1} d_t + N_t c_t + K_{t+1}, \\
K_t = N_{t-1} s_{t-1}, \qquad K_{t+1} &= N_t s_t, \\
I_t &= K_{t+1} - (1-\delta)K_t, \\
c_t &= w_t - s_t, \\
s_t &= s(\omega(k_t), R_{t+1}), \\
d_t &= R_t s_{t-1}.
\end{aligned}
$$

The existence of a temporary equilibrium is guaranteed because the functions are single-valued.

3.7 Intertemporal equilibrium with perfect foresight

The equilibrium equation that links two consecutive periods is the capital accumulation equation. In particular, the savings of young individuals at period t are transformed into productive capital at $t+1$.

Note: in this model, it is useful to write the equations first in aggregate terms and then convert them to its intensive-form representation.

$$
K_{t+1} = N_t s_t = N_t s\left(\omega(k_t), R_{t+1}\right).
$$

In intensive form:

$$
k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{N_t}{N_{t+1}} s\left(\omega(k_t), R_{t+1}\right) = \frac{1}{1+n} s\left(\omega(k_t), R_{t+1}\right).
$$

Finally, we can incorporate capital-market equilibrium and perfect foresight by replacing R_{t+1} with $1-\delta+f'(k_{t+1})$:

$$k_{t+1} = \frac{1}{1+n} s\left(\omega(k_t), 1-\delta+f'(k_{t+1})\right).$$

Intertemporal equilibrium (under perfect foresight): Given an initial capital stock per young worker $k_0 = K_0/N_0$, an intertemporal equilibrium is a sequence of temporary equilibria that satisfies for all $t \geq 0$ the equation:

$$k_{t+1} = \frac{1}{1+n} s\left(\omega(k_t), 1-\delta+f'(k_{t+1})\right).$$

Note: Croix and Michel (2009, 20–27) discusses the existence and uniqueness of the intertemporal equilibrium.

3.7.1 Existence of an intertemporal equilibrium

The existence of at least one next-period equilibrium is guaranteed by the properties of the functions. The proof is involved and is presented below. Having an intertemporal equilibrium means having a solution for k_{t+1} in the equation

$$k_{t+1} = \frac{1}{1+n} s(\omega(k_t), 1-\delta+f'(k_{t+1})),$$

where k_t is predetermined at t.

Proof

For $w > 0$, the proof uses the following inequality for savings:

$$0 < s\left(w, 1-\delta+f'(k)\right) < w.$$

In words, individuals have positive savings but save only part of their total income. If $w = 0$, then $s = 0$ and $k = 0$ is the solution.
Next, define

$$H(k, w) = (1 + n)k - s\left(w, 1 - \delta + f'(k)\right) = 0.$$

Basically, here we are using the definition of an intertemporal equilibrium. Having an intertemporal equilibrium is then equivalent to finding a k satisfying the previous equation.
We use the intermediate value theorem to show that at least one solution exists. First, we analyse the behaviour of $H(k, w)$ when k tends to $+\infty$.
From $0 < s\left(w, 1 - \delta + f'(k)\right) < w$, we have

$$0 < \frac{s(w, 1 - \delta + f'(k))}{k} < \frac{w}{k}.$$

Limit when $k \to +\infty$
Keeping w fixed, the limit of $\frac{w}{k}$ when $k \to +\infty$ is 0. Then,

$$\lim_{k \to +\infty} \frac{s(w, 1 - \delta + f'(k))}{k} = 0.$$

Consequently,

$$\begin{aligned}\lim_{k \to +\infty} \frac{H(k, w)}{k} &= \lim_{k \to +\infty} \left[(1 + n) - \frac{s(w, 1 - \delta + f'(k))}{k}\right] \\ &= 1 + n > 0.\end{aligned}$$

Limit when $k \to 0$
When k tends to zero, we distinguish two cases regarding $f'(k)$ and therefore the gross return. It can be that either

- Case $\lim_{k \to 0} f'(k) = f'(0) > 0$ and finite. Then, the function $s(w, 1 - \delta + f'(k))$ is well defined and positive and we have:

$$\lim_{k \to 0} H(k, w) = \lim_{k \to 0} [(1+n)k - s(w, 1-\delta + f'(k))]$$
$$= -s(w, 1-\delta + f'(0)) < 0.$$

- Case $\lim_{k \to 0} f'(k) = +\infty$. Two things can occur in that case.
  - Savings are positive: $\lim_{k \to 0} s(w, 1-\delta + f'(k)) > 0$. This case is analogous to the previous one:

    $$\lim_{k \to 0} H(k, w) = \lim_{k \to 0} [(1+n)k - s(w, 1-\delta + f'(k))]$$
    $$< 0.$$

  - Savings tend to zero: $\lim_{k \to 0} s(w, 1-\delta + f'(k)) = 0$. In this case, as the gross return goes to infinity, individuals may save an amount tending to zero. This implies that consumption in the second period, d, tends to infinity. To see this, first notice that $d = [1-\delta + f'(k)]s(w, 1-\delta + f'(k))$. Moreover, from the first order conditions we know that:
    $$u'(d) = \frac{u'(w-s)}{\beta[1-\delta + f'(k)]}.$$

    and hence

    $$\lim_{k \to 0} u'(d) = \lim_{k \to 0} \frac{u'(w-s)}{\beta[1-\delta + f'(k)]} = 0.$$

    Hence, $u'(d)$ tends to zero which means that d goes to infinity. Therefore:

    $$\lim_{k \to 0} [1-\delta + f'(k)]s(w, 1-\delta + f'(k)) = +\infty.$$

We also know that $0 < f'(k)k < f(k)$ because $f'(k)k + \omega(k) = f(k)$ and $\omega(k) > 0$. Therefore, for a bounded $0 < k < 1$ we also have $f'(k)k < f(1)$. Hence, for $k < 1$:

$$\begin{aligned}\frac{s(w, 1-\delta+f'(k))}{k} &= \frac{f'(k)s(w, 1-\delta+f'(k))}{f'(k)k} \\ &> \frac{f'(k)s(w, 1-\delta+f'(k))}{f(1)}.\end{aligned}$$

Then,

$$\lim_{k \to 0} \frac{s(w, 1-\delta+f'(k))}{k} = +\infty$$

and

$$\frac{H(k, w)}{k} = 1 + n - \frac{s(w, 1-\delta+f'(k))}{k} < 0$$

for small k.

3.7.2 Uniqueness of the intertemporal equilibrium

Having established that at least one level k_{t+1} exists that solves the equation, we would like it to be unique, thus having a unique equilibrium. Otherwise, for some level k there would be multiple values for k_{t+1} (meaning multiple values for the savings). Individuals lack a coordinating mechanism to select any of these equilibria.

We show that the intertemporal elasticity of substitution σ is important in determining uniqueness. In particular, with CIES utility, $\sigma \geq 1$ is sufficient for uniqueness. We will revisit this issue later.

We have used the intermediate value theorem to show that solutions exist. Moreover, the proof shows that $H(k,w) < 0$ for sufficiently small positive k and $H(k,w) > 0$ for sufficiently large k. Therefore, if $H(k,w)$ is strictly increasing in k, there is exactly one solution to $H(k,w) = 0$. Hence, we would have a unique solution. This means that, given k_t, we can find a unique value k_{t+1}.

If we take the derivative of $H(k,w)$ with respect to k, we obtain:

$$1 + n - s'_R f''(k).$$

Consequently, for H to be strictly increasing it is sufficient to impose:

Assumption OLG 5:

$$1 + n - s'_R f''(k) > 0.$$

And then, $k_{t+1} = g(k_t)$.

In particular, this condition indicates that the effect of the gross return on savings should *not* be too negative.

Moreover, if **OLG 5** is satisfied, then k_{t+1} is increasing in k_t. Indeed,

$$k_{t+1} = \frac{1}{1+n} s(\omega(k_t), R(k_{t+1})).$$

and

$$\frac{\mathrm{d}k_{t+1}}{\mathrm{d}k_t} = \frac{\overbrace{s'_w \omega'(k_t)}^{>0}}{1 + n - s'_R \underbrace{R'(k_{t+1})}_{=f''(k_{t+1})<0}} > 0,$$

and k_{t+1} increases in k_t.

! Important

Hence, if the function H is strictly increasing, the solution is unique. This condition depends on the sign of:

$$\frac{\partial H(k_{t+1}, w_t)}{\partial k_{t+1}} = 1 + n - s'_R f''(k_{t+1}).$$

Finally, we can use a more restrictive —but easier to work with— assumption that ensures a unique solution for the intertemporal equilibrium. Assumption **OLG 5** states:

$$1 + n - s'_R f''(k) > 0.$$

Therefore, it is sufficient to have

$$s'_R \geq 0$$

meaning that savings increase with the gross return. This behaviour is readily verified when the intertemporal elasticity of substitution is at least one. Then, for CIES utility, $\sigma \geq 1$ is sufficient for monotonic dynamics and a unique next-period equilibrium.

We have different cases:

1. $s'_R = 0$, which happens under log-utility. In this case, $s'_R = 0 > \frac{1+n}{R'(k_{t+1})}$ and $H(k_{t+1})$ is always increasing.
2. $s'_R > 0$, the intertemporal elasticity of substitution is greater than 1 and individuals are willing to trade off higher future consumption against present consumption. Savings increase to consume more in the future. Then, clearly $s'_R > 0 > \frac{1+n}{R'(k_{t+1})}$.
3. If $s'_R < 0$, uniqueness can still hold, but it requires the negative return effect on saving not to be too strong.

3.7.3 Possible multiple equilibria and non-monotonic dynamics

When $s'_R < 0$, the sufficient condition for uniqueness may fail; it does not follow that multiplicity must occur. For some utility and production parameters, the implicit equilibrium relation can bend backward and admit several values of k_{t+1} for the same k_t. This section is based on Groth (2016, 92–93).

Such values are alternative, self-fulfilling perfect-foresight equilibria. If all agents coordinate on one anticipated k_{t+1}, its associated return determines saving and asset-market clearing realizes that same capital stock. A low CIES parameter $\sigma < 1$ makes this possibility more likely because saving decreases with the return, but it is neither necessary nor sufficient by itself.

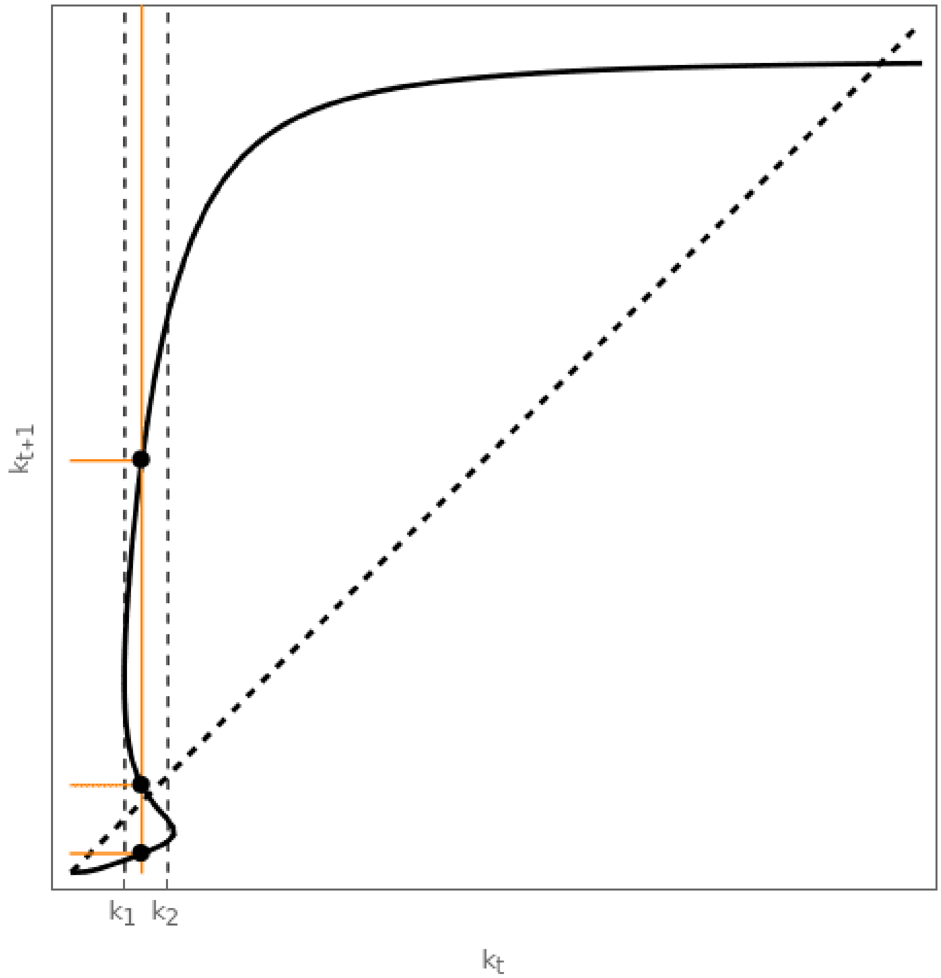

Figure 3.1: Multiple equilibria

The figure illustrates the shape that can generate this multiplicity. Multiplicity means that perfect foresight does not select a unique equilibrium; it does not mean that realized outcomes differ from expectations within a selected equilibrium.

3.8 Steady states

We assume that the intertemporal equilibrium exists and is unique. As noted when characterising uniqueness, for CIES utility, $\sigma \geq 1$ is a sufficient condition. Therefore, the dynamics of capital are given by:

$$k_{t+1} = \frac{1}{1+n} s(\omega(k_t), 1 - \delta + f'(k_{t+1})).$$

Hence, k_{t+1} is an implicit function of k_t:

$$k_{t+1} = g(k_t).$$

The steady states of this economy can be found solving the previous equation for $k_{t+1} = k_t = \bar{k}$.

$$\bar{k} = \frac{1}{1+n} s\left(\omega(\bar{k}), 1 - \delta + f'(\bar{k})\right).$$

3.8.1 Steady state with $\bar{k} = 0$

A steady state with $\bar{k} = 0$ is only possible if $f(0) = 0$ or, equivalently, $\omega(0) = 0$. We may call this steady state an *autarky steady state.* In that case, since $\omega(0) = 0$ and we know that individuals save a fraction of their income:

$$s(\omega(0), R(0)) = s(0, R(0)) = 0.$$

Hence, given zero capital, savings equal zero and capital remains there, constituting a steady state.

Alternatively, if it is possible to produce without capital (CES production function with large enough substitutability), the autarkic steady state does not exist.

3.8.1.1 Example of the autarkic steady state

Take $F(K, L) = (\alpha K^\rho + (1-\alpha)L^\rho)^{\frac{1}{\rho}}$ and $u(c) = \log(c)$. Furthermore, assume a low elasticity of substitution, $\rho << 0$. In intensive form, production equals:

$$f(k_t) = (\alpha k_t^\rho + (1-\alpha))^{\frac{1}{\rho}}.$$

Wages and savings are:

$$\begin{aligned}\omega(k_t) &= (1-\alpha)(\alpha k^\rho + (1-\alpha))^{\frac{1}{\rho}-1} \\ s_t &= \frac{\beta}{1+\beta} w_t.\end{aligned}$$

Then, capital accumulation becomes:

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)\left[\alpha k_t^\rho + 1 - \alpha\right]^{\frac{1-\rho}{\rho}}.$$

With low substitutability, $\rho < 0$, zero is always a possible steady state. There can be two additional, positive steady states depending on the curvature of k_{t+1}.

Similarly, the Cobb-Douglas case $\rho \to 0$ also has zero as a steady state. However, it is unstable.

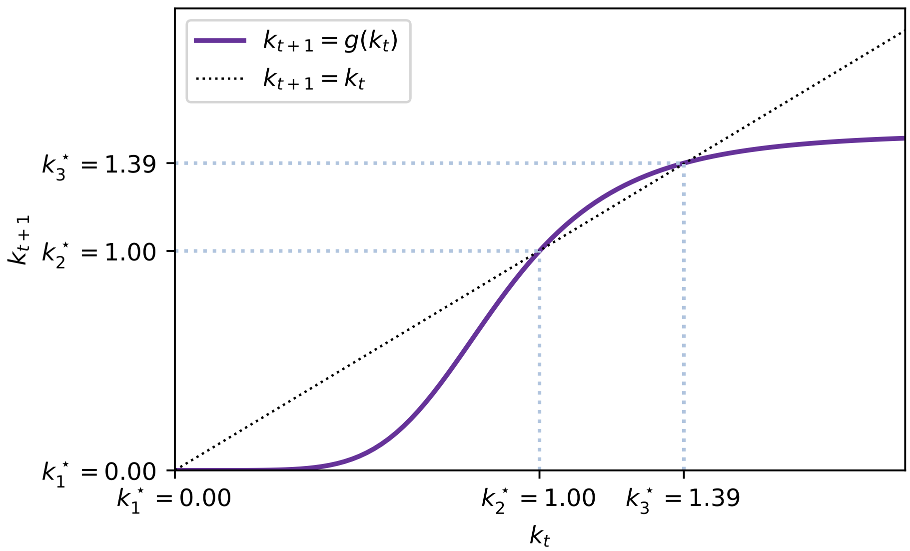

Figure 3.2: Autarky steady state and two positive steady states

3.8.2 Other steady state

The economy can also present interior steady states with $\bar{k} > 0$. Unlike the Ramsey model, here we can have stable and unstable steady states, as we shall see. Moreover, the configuration varies depending on the production and utility functions.

Two steady states, autarky and positive This is a common configuration that arises, for instance, under a Cobb-Douglas production function and log-utility. Other functions can also exhibit this behaviour, particularly those with similar functional forms.

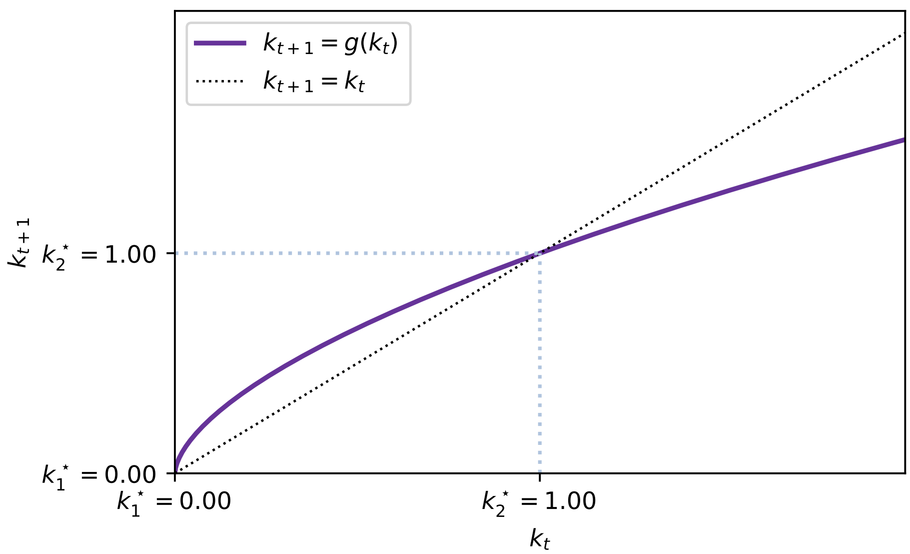

Figure 3.3: Autarky steady state and a positive steady state

Single, non-autarky steady state This steady state is characterised by a unique positive capital level $\bar{k} > 0$. This configuration requires $f(0) > 0$. With a CES production function, capital and labour must be sufficiently substitutable, that is, $0 < \rho < 1$: $(\alpha k^\rho + (1 - \alpha))^{1/\rho}$.

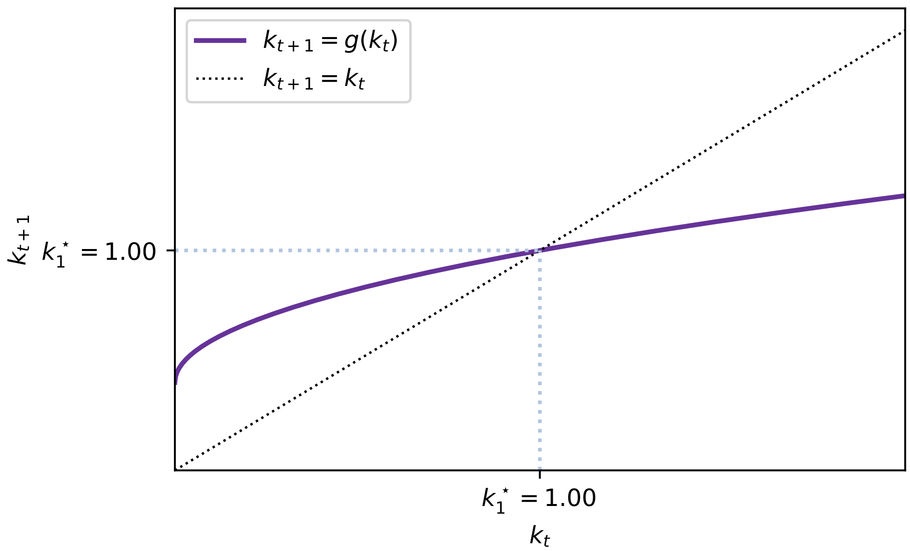

Figure 3.4: Single, non-autarky steady state

Only an autarky steady state In this case, $f(0) = 0$ is required so that the autarky steady state can exist. Moreover, it requires $g(k) < k$ for every $k > 0$. This type of setup can arise with $\rho < 0$, that is, when capital and labour are gross complements and the transition curve is sufficiently low.

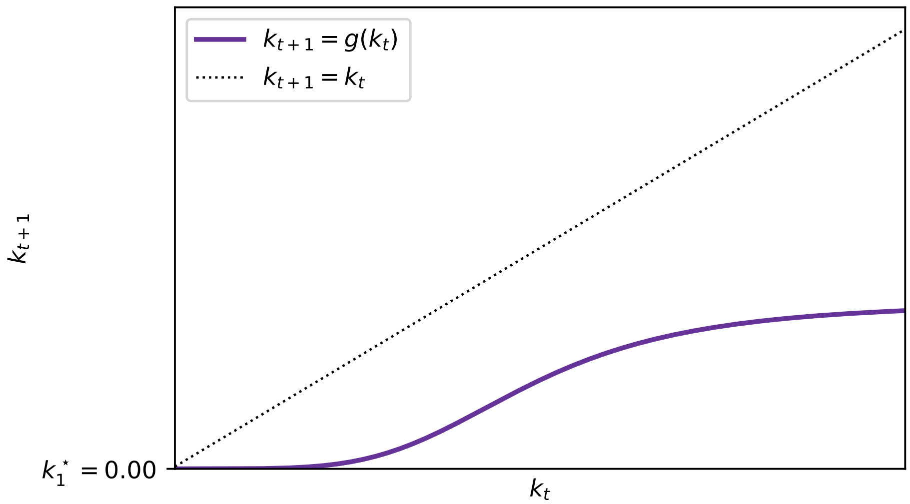

Figure 3.5: Only an autarky steady state

3.9 Stability of the steady state

This model can feature one or several steady states. This depends on the characteristics of the production and utility functions.

Under the sufficient condition used above, g is increasing, so a trajectory cannot cross a fixed point and is monotonic over time. We impose the following condition (Croix and Michel 2009, 25):

$$\forall c > 0, \quad u'(c) + cu''(c) \geq 0 \Leftrightarrow \sigma(c) \geq 1.$$

Because the dynamics are monotonic, capital can converge either to $0, \bar{k} \in (0, +\infty)$, or $+\infty$.

3.9.1 The economy *never* goes to $k = +\infty$

This is equivalent to saying that, starting with a high level of capital k_0, the dynamics imply $k_{t+1} < k_t$.

First, we have that:

$$g(k) = \frac{1}{1+n} s\left(\omega(k), 1 - \delta + f'(\underbrace{g(k)}_{k_{t+1}})\right) < \frac{\omega(k)}{1+n}.$$

This holds because savings s are *always* a fraction of wages ω.

Second, we show that:

$$\lim_{k \to +\infty} \frac{g(k)}{k} = 0.$$

To see the intuition for this limit, assume for a moment it holds. Then, it must be that for large enough k, $g(k) = k_{t+1} < k_t$ and the dynamics are decreasing and converge to some $\bar{k}$. In this case, $\bar{k}$ is the largest steady state, and because the dynamics are monotonic, the economy never goes to the other side of the steady state.

We show next that $\lim_{k \to +\infty} \frac{g(k)}{k} = 0$. We do so by showing that $\lim_{k \to +\infty} \frac{\omega(k)}{k} = 0.$

Proof

We start by noting that:

$$\frac{\omega(k)}{k} = \frac{f(k)}{k} - f'(k) \leftarrow \omega(k) = f(k) - f'(k)k.$$

When $k \to +\infty$, **the first term** $f(k)/k$ admits a finite nonnegative limit, which we call l_1. This is because:

1. $\frac{f(k)}{k} > 0.$
2. $\frac{f(k)}{k}$ is decreasing:

$$\begin{aligned} \frac{\partial[f(k)/k]}{\partial k} &= \frac{f'(k)k - f(k)}{k^2} \\ &= -\frac{f(k) - f'(k)k}{k^2} \\ &= -\frac{\omega(k)}{k^2} < 0. \end{aligned}$$

The **second term** $f'(k)$ is decreasing and positive, thus it admits a limit l_2 when $k \to +\infty$.
Apply the mean value theorem evaluated at k and $2k$:

$$f(2k) - f(k) = (2k - k)f'(k(1+\theta)), \quad \text{with}\, \theta \in (0,1).$$

Then,

$$\frac{2f(2k)}{2k} - \frac{f(k)}{k} = f'(k(1+\theta))$$

Taking the limit when $k \to +\infty$ we obtain $2l_1 - l_1 = l_2 \implies l_1 = l_2$.
Finally,

$$\lim_{k\to+\infty} \frac{\omega(k)}{k} = \lim_{k\to+\infty} \frac{f(k)}{k} - f'(k) = l_1 - l_2 = 0.$$

Since $\lim_{k\to+\infty} \frac{\omega(k)}{k} = 0$ and $\frac{g(k)}{k} < \frac{\omega(k)}{k}$ then $\lim_{k\to+\infty} \frac{g(k)}{k} = 0$. In conclusion, if the initial value of capital is large enough, the dynamics are decreasing and converge to a finite nonnegative steady state. The limit can be zero when zero is the only steady state.

3.10 Examples

3.10.1 Example 1: Log-utility and Cobb-Douglas

Suppose that $u(c, d) = \log(c) + \beta \log(d)$ and $f(k_t) = Ak_t^\alpha$.

The wage rate is given by

$$w_t = f(k_t) - k_t f'(k_t) = (1-\alpha)Ak_t^\alpha$$

and the savings function is obtained solving

$$u'(w_t - s_t) = \beta R_{t+1} u'(R_{t+1} s_t).$$

Rearranging and isolating s we are left with the savings function

$$s_t = \frac{\beta}{1+\beta} w_t.$$

The dynamics are given by the capital accumulation law

$$k_{t+1} = \frac{1}{1+n} s_t = \frac{1}{1+n}\frac{\beta}{1+\beta} w_t = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)Ak_t^\alpha.$$

The model with log-utility and Cobb-Douglas production functions has two steady states: zero and a unique positive steady state. We can solve for them:

$$
\begin{aligned}
\phi &\equiv \frac{\beta(1-\alpha)A}{(1+n)(1+\beta)}, \\
\bar{k} &= \phi \bar{k}^{\alpha} \\
\implies \bar{k} &\in \left\{0,\ \phi^{\frac{1}{1-\alpha}}\right\}.
\end{aligned}
$$

The dynamics are depicted in Figure 3.3.

Local dynamics: stability

We check the stability of the two steady states using the first derivative evaluated at the steady state. **Note:** in general we would use the Jacobian matrix, but the OLG model has only one equation in one variable.

Remember that a fixed point of a one-dimensional discrete-time system is locally asymptotically stable if $|g'(\bar{k})| < 1$ and unstable if $|g'(\bar{k})| > 1$. It is hyperbolic when $|g'(\bar{k})| \neq 1$; the equality case requires further analysis.

In our case:

$$
\begin{aligned}
g(k) = k_{t+1} &= \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)Ak_t^{\alpha} \\
g'(k) &= \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)\alpha Ak_t^{\alpha-1} > 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
\bar{k} = 0 &\implies g'(0) = +\infty \qquad \to \text{unstable} \\
\bar{k} = \phi^{\frac{1}{1-\alpha}} &\implies g'(\phi^{\frac{1}{1-\alpha}}) = \phi\alpha\phi^{\frac{\alpha-1}{1-\alpha}} = \alpha \in (0,1) \qquad \to \text{stable.}
\end{aligned}
$$

Only an autarky steady state In this case, $f(0) = 0$ is required so that the autarky steady state can exist. Moreover, it requires $g(k) < k$ for every $k > 0$. This type of setup can arise with $\rho < 0$, that is, when capital and labour are gross complements and the transition curve is sufficiently low.

3.10.2 Log-utility and CES technology

Under a CES specification, $y_t = A[\alpha k_t^\rho + (1-\alpha)]^{1/\rho}$, with $\rho < 1$ and $\rho \neq 0$, wages are given by

$$\omega_t = A(1-\alpha)\left[\alpha k_t^\rho + 1 - \alpha\right]^{\frac{1-\rho}{\rho}} .$$

Hence, the dynamics of capital are governed by:

$$\begin{aligned} k_{t+1} = g(k_t) &= \frac{1}{1+n} s_t \\ &= \frac{\beta A(1-\alpha)}{(1+n)(1+\beta)}\left[\alpha k_t^\rho + 1 - \alpha\right]^{\frac{1-\rho}{\rho}} . \end{aligned}$$

The curve $g(k)$ is increasing:

$$g'(k) = \frac{\beta A(1-\alpha)}{(1+n)(1+\beta)} \alpha(1-\rho)\left[\alpha k^\rho + 1 - \alpha\right]^{\frac{1-2\rho}{\rho}} k^{\rho-1} > 0.$$

Depending on the value of ρ we can have different configurations regarding the steady states. Ultimately, what matters is the concavity of the function $g(k_t)$.

$$\begin{aligned} g''(k_t) = &-\frac{\beta A(1-\alpha)(1-\rho)\alpha}{(1+n)(1+\beta)} \\ &\times \left[\alpha k^\rho + 1 - \alpha\right]^{\frac{1-3\rho}{\rho}} k^{\rho-2} \\ &\times \left[\rho\alpha k^\rho + (1-\alpha)(1-\rho)\right] . \end{aligned}$$

3.10.2.1 Unique steady state, $\bar{k} > 0$

If $\rho \in (0, 1)$, the second derivative is clearly negative, and g is concave. Moreover, $g(0) > 0$, so $\bar{k} = 0$ cannot be a steady state:

$$\begin{aligned} g(k_t) &= \frac{\beta A(1-\alpha)}{(1+n)(1+\beta)}(\alpha k^\rho + 1 - \alpha)^{\frac{1-\rho}{\rho}}, \\ \implies g(0) &= \frac{\beta A(1-\alpha)^{1/\rho}}{(1+n)(1+\beta)} > 0. \end{aligned}$$

In that case, the economy displays a unique steady state, and it is globally stable.

See Figure 3.4.

3.10.2.2 Unique $\bar{k} = 0$ or multiple steady states

If $\rho < 0$, the function $g(k)$ changes concavity. In particular, there exists a level $\hat{k}$ such that:

$$\begin{aligned} g''(k) > 0 \,\text{if}\, k < \hat{k}. \\ g''(k) < 0 \,\text{if}\, k > \hat{k}. \end{aligned}$$

with

$$\hat{k} = \left(-\frac{(1-\alpha)(1-\rho)}{\rho\alpha}\right)^{\frac{1}{\rho}} > 0 \,\text{since}\, \rho < 0.$$

Moreover, $g(0) = 0$ so $\bar{k} = 0$ can be a steady state.

In this case, there are two sub-cases:

1. If $g(k) < k$ for every $k > 0$, equivalently $\frac{\omega(k)}{k} < (1+n)\frac{1+\beta}{\beta}$ for every $k > 0$, then the transition curve is always below the 45-degree line. Then, there is one unique steady state: $\bar{k} = 0$. The economy

converges towards this unique steady state. We can derive the condition above from:

$$k_{t+1} < k_t \implies \frac{1}{1+n} s\left(\omega(k_t), R_{t+1}\right) < k_t \implies$$
$$\frac{1}{1+n}\frac{\beta}{1+\beta} w_t < k_t \implies \frac{w_t}{k_t} < (1+n)\frac{1+\beta}{\beta}.$$

2. If $g(k) > k$ for some $k > 0$, equivalently $\frac{\omega(k)}{k} > (1+n)\frac{1+\beta}{\beta}$ there, then the S-shaped transition curve crosses the 45-degree line twice at positive capital levels $\bar{k}_a < \bar{k}_b$, in addition to the zero steady state.
    1. All trajectories starting at $k_0 < \bar{k}_a$ converge to $\bar{k} = 0$ and $\bar{k} = 0$ is locally stable in the range $[0, \bar{k}_a]$. The steady state with $\bar{k} = 0$ is a **poverty trap**: whenever the economy starts with a low enough level of capital, it will always converge towards the $\bar{k} = 0$ steady state.
    2. Trajectories starting at $k_0 = \bar{k}_a$ remain at $\bar{k}_a$. This steady state is unstable.
    3. Finally, trajectories starting with $k_0 > \bar{k}_a$ converge towards $\bar{k}_b$, which is a locally stable steady state in $(\bar{k}_a, +\infty)$.

See Figure 3.5 and Figure 3.2.

3.11 Solved example

Suppose that production can be parametrized using the following Cobb-Douglas specification:

$$Y = F\left(K_t, L_t\right) = K_t^{\frac{1}{2}} L_t^{\frac{1}{2}}.$$

Furthermore, assume full depreciation, $\delta = 1$, and the following period utility function:

$$u(c_t) = \frac{c_t^{1-\frac{1}{3}} - 1}{1 - \frac{1}{3}}$$

where the intertemporal elasticity of substitution is 3.

We can compute the production function, wage, and capital rental rate in intensive form as:

$$\begin{aligned} f(k_t) &= k_t^{1/2} \\ w_t &= \frac{1}{2} k_t^{1/2} \\ r_t &= \frac{1}{2} k_t^{-1/2} \end{aligned}$$

Similarly, the savings function can be derived from the Euler equation as follows:

$$\begin{aligned} u'(w_t - s_t) &= \beta R_{t+1} u'(s_t R_{t+1}) \implies \\ \implies (w_t - s_t)^{\frac{-1}{3}} &= \beta R_{t+1} (R_{t+1} s_t)^{\frac{-1}{3}} \implies \\ \implies (w_t - s_t) &= \beta^{-3} R_{t+1}^{-2} s_t \implies \\ s_t &= \frac{w_t}{1 + \beta^{-3} R_{t+1}^{-2}} \end{aligned}$$

Therefore, the capital accumulation equation is:

$$
\begin{aligned}
k_{t+1} &= \frac{1}{1+n}\frac{w_t}{1+\beta^{-3}R_{t+1}^{-2}},\\
w_t &= \frac{1}{2}k_t^{1/2}, \qquad R_{t+1} = \frac{1}{2}k_{t+1}^{-1/2},\\
\implies k_{t+1} &= \frac{1}{1+n}\frac{\frac{1}{2}k_t^{1/2}}{1+4\beta^{-3}k_{t+1}},\\
\frac{1}{2}k_t^{1/2} &= (1+n)k_{t+1}\left(1+4\beta^{-3}k_{t+1}\right),\\
k_{t+1} &= \frac{-(1+n)+\sqrt{(1+n)^2+8(1+n)\beta^{-3}k_t^{1/2}}}{8(1+n)\beta^{-3}}.
\end{aligned}
$$

We can then show that the autarky steady state exists. Indeed, because we have $f(0) = 0$, then $\bar{k} = 0$ is a steady state. The economy has a second steady state. It is most conveniently characterised by a third-degree polynomial. Nevertheless, let's sketch the process, knowing that a steady state solves $k_t = k_{t+1} = \bar{k}$. To simplify the notation, in what follows I will write k instead of $\bar{k}$.

$$
4\beta^{-3}k^2 + k - \frac{k^{1/2}}{2(1+n)} = 0.
$$

Letting $x = k^{1/2} \geq 0$ gives

$$
x\left(4\beta^{-3}x^3 + x - \frac{1}{2(1+n)}\right) = 0.
$$

Thus, $k = 0$ is a steady state. The expression in parentheses is strictly increasing in x, is negative at zero, and tends to infinity. It therefore has exactly one positive root, which gives exactly one positive steady state.

The derivative of the dynamic function with respect to k_t is:

$$\frac{\partial k_{t+1}}{\partial k_t} = \frac{1}{4\sqrt{k_t}\sqrt{(1+n)\left(1+n+\frac{8\sqrt{k_t}}{\beta^3}\right)}}$$

At $k_t = k_{t+1} = 0$, the right derivative is $+\infty$, so the zero steady state is unstable. The derivative need not be below one at every positive k_t, but at the positive fixed point the steady-state equation implies

$$g'(\bar{k}) = \frac{4\beta^{-3}\bar{k}+1}{2(8\beta^{-3}\bar{k}+1)} \in \left(\frac{1}{4}, \frac{1}{2}\right).$$

The unique positive steady state is therefore locally stable. In fact, the plot corresponds to Figure 3.3.

3.12 The golden-rule level of capital

The first welfare theorem states that a competitive equilibrium is Pareto optimal. However, the first welfare theorem has two requisites the Diamond model does not meet: a finite number of goods and a finite number of agents. Therefore, the allocation resulting from the competitive equilibrium may *not* be Pareto optimal. This section is based on Croix and Michel (2009, Ch. 2) and Groth (2016).

In the Diamond model —and different from the Ramsey model—, the capital stock on the balanced growth path may exceed the golden-rule level. This implies that a permanent increase in consumption is possible.

When discussing the golden-rule level of capital, we are *not* interested in how consumption is divided between the old and the young. Rather, we seek to maximise aggregate consumption that can be sustained period after period. This criterion determines the efficient capital stock, but not how consumption should be allocated between the young and old; that distribution requires a separate welfare criterion. First, let $\mathfrak{C}_t$ represent total consumption:

$$\mathfrak{C}_t \equiv C_t + D_t.$$

The economy-wide resource constraint dictates that total production is either consumed or invested:

$$\mathfrak{C}_t + K_{t+1} = F(K_t, N_t) + (1-\delta)K_t.$$

Alternatively, aggregate consumption per unit of labour is:

$$\begin{aligned}\mathfrak{c}_t \equiv \frac{\mathfrak{C}_t}{N_t} = \frac{F(K_t, N_t) + (1-\delta)K_t - K_{t+1}}{N_t} = \\ = f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}.\end{aligned}$$

The *golden-rule level of the capital-labour ratio* $k^{\mathcal{GR}}$ is the value of the capital-labour ratio k that results in the highest possible *sustainable* level of consumption per unit of labour. The fact that it is sustainable requires that it is replicable forever. For this reason, we consider the steady state with $k_{t+1} = k_t = \bar{k}$. The resource constraint at the steady state simplifies to:

$$\bar{\mathfrak{c}} = f(\bar{k}) - (\delta + n)\bar{k} \equiv \mathfrak{c}(k).$$

We maximise this function with respect to $\bar{k}$:

$$\mathfrak{c}'(\bar{k}) = f'(\bar{k}) - (n+\delta) = 0$$

The fact that $\mathfrak{c}''(\bar{k}) = f''(\bar{k}) < 0$ assures that we have the maximum.

Assuming that $n + \delta > 0$ and that f satisfies:

$$\lim_{k\to+\infty} f'(k) < n+\delta < \lim_{k\to 0} f'(k),$$

then $\mathfrak{c}'(\bar{k}) = f'(\bar{k}) - (n+\delta) = 0$ has a solution in $\bar{k}$ and it is unique. Hence, the golden-rule level of capital is given by:

$$f'(k^{\mathcal{GR}}) = n+\delta.$$

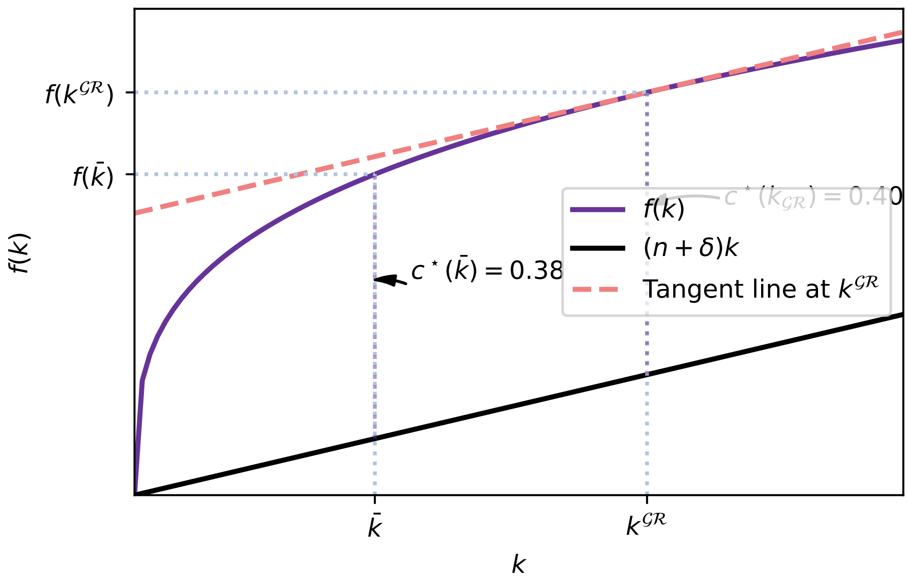

Figure 3.6: The golden-rule level of capital

The highest sustainable consumption level per unit of labour is obtained when, at steady state, the net marginal productivity of capital equals the growth rate of the economy.

3.12.1 Over- and under-accumulation of capital

The golden-rule level of capital is given by $f'(k) = n + \delta$. However, in the decentralised economy, the steady-state level of savings is highly *unlikely* to coincide with it. For instance, with log utility and $f(k) = Ak^{\alpha}$, the decentralized positive steady state satisfies

$$f'(\bar{k}) = \frac{\alpha(1+n)(1+\beta)}{\beta(1-\alpha)},$$

which generally differs from $n + \delta$.

If the steady-state level of the economy is such that $f'(\bar{k}) < n + \delta$, the economy has accumulated too much capital: if $f'(\bar{k}) < f'(k^{\mathcal{GR}}) \implies \bar{k} > k^{\mathcal{GR}}$. Conversely, if the marginal product at the competitive steady state satisfies $f'(\bar{k}) > n + \delta$, then the economy lacks capital relative to the golden rule.

3.12.1.1 Improving the situation: dynamic inefficiency

Suppose that the decentralised economy reaches a situation of overaccumulation of capital, that is, $f'(\bar{k}) < n + \delta$. It is possible to improve upon it by means of redistribution. Simply, a planner could order young individuals to save less, as to reach $k^{\mathcal{GR}}$. Doing so increases the resources available for consumption per worker, which can be allocated between the old and young to support a Pareto improvement. In subsequent periods, the economy would remain at $k^{\mathcal{GR}}$, which maximises sustainable aggregate consumption per worker.

In particular, assume we impose a lump-sum tax on all young individuals and we transfer it to the old generation. The transfer is fully consumed (old individuals gain utility from consuming it). Suppose the transfer is one good from each young to the old. There are $1 + n$ young individuals per old individual, so the old receive $1+n$ goods. Let's repeat this scheme every period.

- The young transfer one good to the old
- Consumption of the young does not change because they refrain from saving.
- In the future, they receive $1 + n$, which is the gross return on their implicit PAYG contribution.
- In the competitive market, they would have obtained the gross return $R = 1 - \delta + f'(k)$.
- Under overaccumulation, $f'(k) < n + \delta$ is equivalent to $R < 1 + n$, so the PAYG gross return is larger.

At the same time, old individuals are also better off. In the first period, when the reform is introduced, they consume more. Later generations obtain a higher return for the tax they pay, leaving them better off, too.

When organising such scheme is possible, we say that the competitive economy was **dynamically inefficient.**

Note: it is possible to solve the dynamic inefficiency introducing money. Young individuals would exchange goods for paper notes with the expectation of exchanging them back again when old.

3.12.1.2 Dynamic efficiency

In the opposite case, when $\bar{k} < k^{\mathrm{GR}}$, reaching the golden-rule level of capital requires increasing it. Suppose we are at time t_0. Increasing the capital to $k^{\mathcal{GR}}$ would certainly improve consumption for all generations $t > t_0$. However, young individuals at t_0 should forgo some consumption to save more and build the necessary capital. Hence, the consumption of the young generation at t_0 necessarily decreases. Our policy of improving utility for all individuals fails in this case.

We call such situations **dynamically efficient**: it is impossible to increase the utility of all generations.

3.12.2 Example

We use a log-utility and Cobb-Douglas case to illustrate over- and under-accumulation of capital.

$$U(c,d) = \log(c) + \beta \log(d)$$
$$f(k) = Ak^{\alpha}.$$

The golden-rule level of capital dictates that total consumption per worker should be maximised. Using the resource constraint:

$$\max \mathfrak{c}_t = f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}.$$

Evaluating it at the steady state yields:

$$\max \mathfrak{c} = f(k) - (n+\delta)k \implies f'(k^{\mathcal{GR}}) = n+\delta.$$

Hence, in our case:

$$A\alpha {k^{\mathcal{GR}}}^{\alpha-1} = n+\delta \implies k^{\mathcal{GR}} = \left(\frac{A\alpha}{n+\delta}\right)^{\frac{1}{1-\alpha}}.$$

In the decentralised economy, we have that:

$$k_{t+1} = \frac{1}{1+n} s\left(\omega(k_t), 1-\delta+f'(k_{t+1})\right) = \frac{1}{1+n}\frac{\beta}{1+\beta}A(1-\alpha)k_t^{\alpha}.$$

Hence, the steady-state level of capital is:

$$\bar{k} = \frac{A\beta(1-\alpha)}{(1+n)(1+\beta)}\bar{k}^{\alpha} \implies \bar{k} = \left[\frac{A\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}.$$

We have capital over-accumulation if: $\bar{k} > k^{\mathcal{GR}} \implies \frac{(n+\delta)\beta}{(1+n)(1+\beta)} > \frac{\alpha}{1-\alpha}$.

Check Romer (2018, 89–90) for an empirical discussion about the dynamic efficiency.

3.13 The Central Planner

Suppose now, instead, that a central planner is in charge of organising consumption and investment for all individuals. The planner operates by aggregating individual utility, discounting future generations at the rate γ. **Note:** γ is the discount rate of future *generations*, not how young people discount old-age utility.

Her utility considers the utility of all generations, including the initial old generation, which owns the initial aggregate capital stock K_0 (equivalently, each initial old agent owns $(1+n)k_0$).

3.13.1 Planner's utility

Planner's utility reads:

$$\sum_{t=-1}^{\infty} \gamma^t U(c_t, d_{t+1}).$$

Note that, although the planner can decide any allocation, she must respect the resource constraint:

$$f(k_t) + (1-\delta)k_t = c_t + \frac{1}{1+n}d_t + (1+n)k_{t+1}.$$

Assume that $U(c_t, d_{t+1})$ is separable: $U(c_t, d_{t+1}) = u(c_t) + \beta u(d_{t+1})$. Expanding the utility of the planner, we can reformulate it in more convenient terms:

$$\begin{aligned}
\sum_{t=-1}^{\infty} \gamma^t \left[u(c_t) + \beta u(d_{t+1})\right] &= \gamma^{-1}u(c_{-1}) + \gamma^{-1}\beta u(d_0) \\
&\quad + u(c_0) + \beta u(d_1) \\
&\quad + \gamma u(c_1) + \gamma\beta u(d_2) + ... \\
&= \sum_{t=0}^{\infty} \gamma^t \left[u(c_t) + \frac{\beta}{\gamma}u(d_t)\right] \\
&\quad + \gamma^{-1}u(c_{-1}).
\end{aligned}$$

The term $\gamma^{-1}u(c_{-1})$ represents the consumption of the generation born at $t = -1$, but since it is a constant it will not affect the maximisation.

Hence, the planner's problem is now how to allocate consumption between the young and old that are alive during period t.

3.13.2 Maximisation

We use a substitution to obtain the planner's optimal allocation, namely, the Euler equation;

$$\begin{aligned}
&\max_{k_{t+1}, c_t, d_t} \sum_{t=0}^{\infty} \gamma^t \left[u(c_t) + \frac{\beta}{\gamma}u(d_t)\right] \\
&\text{s.t.} \quad f(k_t) + (1-\delta)k_t = c_t + \frac{d_t}{1+n} + (1+n)k_{t+1}.
\end{aligned}$$

$$
\begin{aligned}
&\max_{c_t, k_{t+1}} \quad \sum_{t=0}^{\infty} \gamma^t \left[u(c_t) + \frac{\beta}{\gamma} u(d_t) \right] \\
&\text{s.t.} \quad d_t = (1+n) \left[f(k_t) + (1-\delta) k_t - (1+n) k_{t+1} - c_t \right].
\end{aligned}
$$

Taking derivatives and equating them to zero yields:

$$
\begin{aligned}
\gamma^t u'(c_t) &= \gamma^t \frac{\beta}{\gamma} u'(d_t)(1+n) \\
\gamma^t \frac{\beta}{\gamma} u'(d_t)(1+n)^2 &= \gamma^{t+1} \frac{\beta}{\gamma} (1+n) \\
&\quad \times u'(d_{t+1}) \left[f'(k_{t+1}) + 1 - \delta \right].
\end{aligned}
$$

Combining both equations we get the Euler equation:

$$
u'(c_t) = \beta u'(d_{t+1}) \left(f'(k_{t+1}) + 1 - \delta \right).
$$

The planner's Euler equation coincides with the decentralised one, where $R_{t+1} = 1 - \delta + f'(k_{t+1})$.

However, the planner also allocates consumption between the young and the old at time t.

$$
\gamma^t u'(c_t) = \gamma^t \frac{\beta}{\gamma} u'(d_t)(1+n)
$$

In the centralised equilibrium, the planner also trades off the consumption of different cohorts alive at time t. This condition has no decentralized counterpart because finitely lived individuals value only their own life-cycle consumption; an old individual does not value the contemporaneous young cohort's consumption.

3.13.3 Steady state and modified golden rule

Using the fact that

$$\gamma^t \frac{\beta}{\gamma} u'(d_t)(1+n)^2 = \gamma^{t+1} \frac{\beta}{\gamma}(1+n) \times u'(d_{t+1}) \left[f'(k_{t+1}) + 1 - \delta\right]$$

we can easily characterise the steady state knowing that $d_t = d_{t+1} = \bar{d}, k_t = k_{t+1} = \bar{k}$.

$$f'(\bar{k}) = \frac{1+n}{\gamma} - 1 + \delta.$$

This equation provides us with the modified golden rule: the level of capital that maximises the planner's utility. Clearly, if $\gamma = 1$, the planner attributes the same weight to all generations and we recover the golden rule: $f'(k) = n + \delta$. As we discussed before, it is quite *unlikely* the decentralised equilibrium converges towards the golden rule (modified or not).

4 de la Croix and Dottori (2008)

This section discusses Croix and Dottori (2008), using a simplified model to illustrate the key points.

4.1 Introduction

This paper analyses the very different population patterns of two remote islands in the Pacific Ocean: Easter Island and Tikopia. While Tikopians managed to control population growth and natural-resource use, the inhabitants of Easter Island engaged in clan competition for control of resources, leading to overpopulation and overexploitation of resources.

A key aspect of this paper is the special role of fertility. Most papers assume that parents derive some utility from having children. However, in de la Croix and Dottori, fertility is chosen strategically before a Nash bargaining process over the control of resources. In particular, a larger population, facilitated by having more children, raises the value of the fallback option during the negotiation process. Consequently, individuals optimally decide to have more children, because this implies a better bargaining position. In this sense, equilibrium fertility rates result from the complementarities between different groups' fertility decisions. In the paper's general model, the fertility externality can generate a population race, exhaust natural resources and produce an environmental collapse. The special case studied below is deliberately simpler: its clan populations converge to a finite common level, so it illustrates strategic convergence rather than population explosion.

4.1.1 Historical data

Based on the archaeological estimates reported in the paper, Easter Island's population increased very little between 400 CE (about 100 people) and 1100 CE. It then grew rapidly, reaching about 10,000 people during 1400–1600 CE. The effects of the population race were apparent by 1600 CE: food consumption declined and the population fell sharply during the seventeenth century. By 1722, when Europeans arrived at Easter Island, the total population was around 3,000 people. In parallel, evidence about Easter Island's forests indicates that tree cutting began after the first settlers arrived around 400 CE. By 1400 CE, deforestation had reached its peak, and when Europeans arrived there were almost no trees on the island.

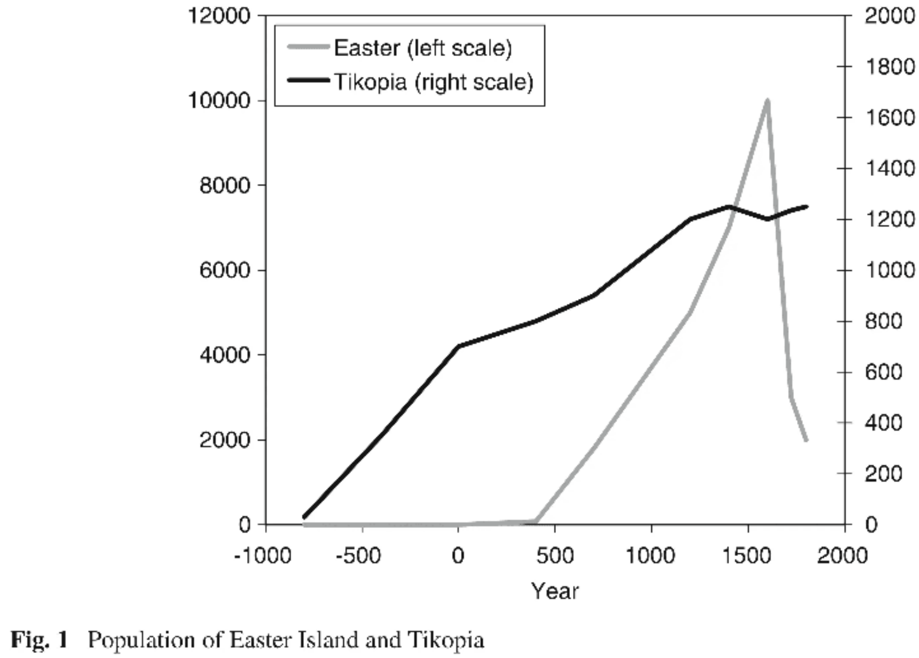

Figure 4.1: Figure 1 in de la Croix and Dottori, 2008

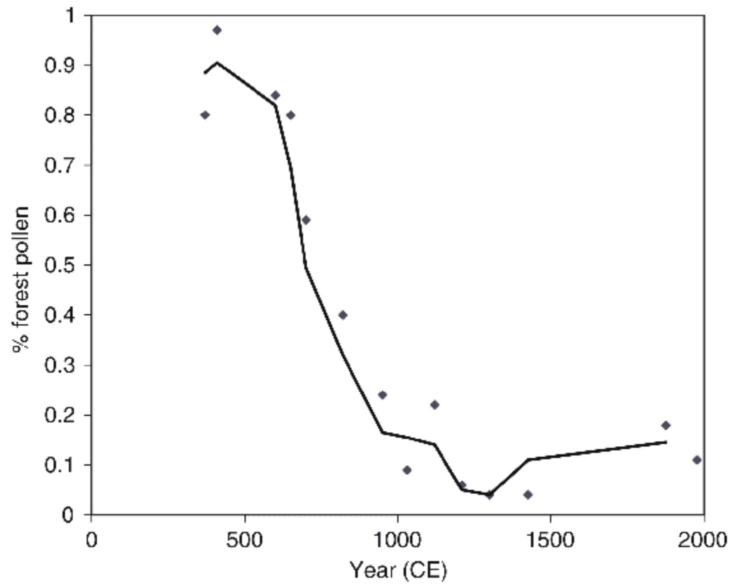

Fig. 2 Forest coverage on Easter Island

Figure 4.2: Figure 2 in de la Croix and Dottori, 2008

Meanwhile, Tikopia was settled around 900 BCE, and its inhabitants practised slash-and-burn agriculture. By 100 BCE, diminishing returns from natural resources led to pig breeding. This lasted until the seventeenth century, when Tikopians abandoned it because pigs required too many resources. The total population stabilised at around 1,200 people and was kept at that level through deliberate mechanisms, including celibacy, abortion, infanticide and sea voyages by young men.

4.2 The model

De la Croix and Dottori use an OLG framework. In the paper, agents live for two periods. However, important decisions are taken at the clan level, which acts as a representative agent. Clans (and individuals) are rational, have perfect foresight and take the actions of the other clan as given. The timing is as follows:

1. Each clan chooses its fertility level.
2. A Nash-Cournot fertility equilibrium arises.
3. Crops are cultivated and shared between clans following a non-cooperative bargaining process.

For simplicity, the island is populated by two opposing clans; every individual belongs to exactly one clan and cannot change clans.

The paper's general model contains four parameters that matter for the population race. Conflict success depends on relative clan size according to

$$\pi_t = \frac{N_{1,t}^{\mu}}{N_{1,t}^{\mu} + N_{2,t}^{\mu}},$$

where $\mu > 0$ measures how responsive Group 1's probability of victory is to relative numbers. Aggregate production is

$$Y_t = A(R_t)L^{\alpha}(N_{1,t} + N_{2,t})^{1-\alpha},$$

where L is fixed land and $\alpha \in (0,1]$ is the land share. War destroys a fraction $\omega \in [0,1]$ of the crop, so the disagreement shares of the original crop are $(1-\omega)\pi_t$ for Group 1 and $(1-\omega)(1-\pi_t)$ for Group 2. Raising children also has a current cost $\lambda n_{i,t}$, with $\lambda \geq 0$; in the general budget, young consumption is

$$c_{i,t} = \left(1 - \frac{\tau}{1 + n_{i,t-1}} - \lambda n_{i,t}\right) y_{i,t}.$$

The Nash bargaining solution therefore assigns Group 1 the crop share

$$\theta_t = \gamma\omega + (1 - \omega)\pi_t,$$

where $\gamma \in (0, 1)$ is Group 1's Nash bargaining weight.

To obtain transparent closed-form dynamics, the remainder of this section imposes the following special case before using population shares, linear production and the cube-root fertility rules:

$$\mu = 1, \qquad \omega = 0, \qquad \lambda = 0, \qquad \alpha = 1. \qquad \text{(Assumption 1)}$$

Thus π_t is Group 1's population share, $Y_t = A(R_t)L$, and $\theta_t = \pi_t$. These restrictions are stronger than the assumptions behind the paper's general environmental-collapse result.

4.2.1 Preferences

Clan i at time t consists of $N_{i,t}$ adults. Adults work, support their parents and have children. Old agents only consume what their children provide for them.[1] Total utility is given by:

$$U_{i,t} = c_{i,t} + \beta d_{i,t+1},$$

where $c_{i,t}$ and $d_{i,t+1}$ represent consumption when young and old, respectively.

4.2.2 Budget

The income of an adult agent is $y_{i,t}$. Each adult supports their parents by giving them some resources. Support for parents depends on the number

[1] Bisin and Verdier also model how parental indoctrination affects the evolution of cultural traits.

of siblings. In particular, each sibling contributes the following share of their income:

$$\frac{\tau}{1 + n_{i,t-1}},$$

where $\tau \in (0, 1)$. Clearly, the contribution decreases with the number of siblings.

Consequently, an agent who had $n_{i,t}$ children receives the following **total** support in old age:

$$d_{i,t+1} = n_{i,t} \frac{\tau}{1 + n_{i,t}} y_{i,t+1}.$$

Under Assumption 1, $\lambda = 0$. Since income is divided between consumption and support for parents, $y_{i,t} = c_{i,t} + \frac{\tau}{1+n_{i,t-1}} y_{i,t}$, consumption when young is therefore

$$c_{i,t} = \left(1 - \frac{\tau}{1 + n_{i,t-1}}\right) y_{i,t}.$$

4.2.3 Population

The population of each clan evolves according to the chosen fertility level:

$$N_{i,t+1} = N_{i,t} n_{i,t}.$$

4.2.4 Production

Under Assumption 1, labour has a zero exponent, land is fixed at L, and total factor productivity depends on the available natural resources R_t. The general production function above consequently reduces to

$$Y_t = A(R_t) L.$$

The dynamics of resources follow the paper by Matsumoto (2002):

$$R_{t+1} = \left(1 + \delta - \delta \frac{R_t}{K} - b(N_{1,t} + N_{2,t})\right) R_t,$$

where $K > 0$ is the carrying capacity, $\delta > 0$ is the intrinsic growth rate of resources and $b > 0$ measures population pressure on resources.

4.2.4.1 Crop-sharing

We denote by θ_t the share of the crop Y_t that Group 1 receives. Therefore, each adult in Groups 1 and 2 obtains:

$$y_{1,t} = \theta_t \frac{Y_t}{N_{1,t}},$$

$$y_{2,t} = (1 - \theta_t) \frac{Y_t}{N_{2,t}}.$$

There are no property rights on the island, and the groups bargain over how to split total production Y_t. If no agreement is reached, the clans fight over the crop.

4.3 Bargaining

Bargaining uses the Nash solution. For $\omega > 0$, it maximises

$$(U_{1,t} - \bar{U}_{1,t})^{\gamma} (U_{2,t} - \bar{U}_{2,t})^{1-\gamma},$$

where $U_{i,t}$ is Group i's utility under agreement and $\bar{U}_{i,t}$ is its disagreement utility. Under Assumption 1, the probability that Group 1 wins is its population share:

$$\pi_t = \frac{N_{1,t}}{N_{1,t} + N_{2,t}}.$$

Thus a larger clan is more likely to win a war. Increasing population also improves the clan's bargaining position by raising its disagreement utility.

To derive the general sharing rule, temporarily retain ω and λ. Suppose the clans agree on a crop share θ_t: Group 1 receives θ_t, and Group 2 receives the remaining $1 - \theta_t$. Indirect utilities are then

$$U_{1,t} = \left(1 - \frac{\tau}{1 + n_{1,t-1}} - \lambda n_{1,t}\right) \frac{\theta_t Y_t}{N_{1,t}} + \beta \frac{n_{1,t}\tau}{1 + n_{1,t}} \frac{\theta_{t+1} Y_{t+1}}{N_{1,t+1}},$$

$$U_{2,t} = \left(1 - \frac{\tau}{1 + n_{2,t-1}} - \lambda n_{2,t}\right) \frac{(1 - \theta_t) Y_t}{N_{2,t}} + \beta \frac{n_{2,t}\tau}{1 + n_{2,t}} \frac{(1 - \theta_{t+1}) Y_{t+1}}{N_{2,t+1}}.$$

In the general model, war destroys the fraction ω of current production. The disagreement shares are therefore $(1-\omega)\pi_t$ and $(1-\omega)(1-\pi_t)$. The current-period parts of the disagreement utilities are

$$\bar{U}_{1,t} = (1 - \omega)\pi_t \left(1 - \frac{\tau}{1 + n_{1,t-1}} - \lambda n_{1,t}\right) \frac{Y_t}{N_{1,t}} + \beta \frac{n_{1,t}\tau}{1 + n_{1,t}} \frac{\theta_{t+1} Y_{t+1}}{N_{1,t+1}},$$

$$\bar{U}_{2,t} = (1-\omega)(1-\pi_t) \left(1 - \frac{\tau}{1 + n_{2,t-1}} - \lambda n_{2,t}\right) \frac{Y_t}{N_{2,t}} + \beta \frac{n_{2,t}\tau}{1 + n_{2,t}} \frac{(1 - \theta_{t+1}) Y_{t+1}}{N_{2,t+1}}.$$

The agreement surpluses that depend on θ_t are proportional to

$$U_{1,t} - \bar{U}_{1,t} = \left(1 - \frac{\tau}{1 + n_{1,t-1}} - \lambda n_{1,t}\right) \left[\theta_t - (1 - \omega)\pi_t\right] \frac{Y_t}{N_{1,t}},$$

$$U_{2,t} - \bar{U}_{2,t} = \left(1 - \frac{\tau}{1+n_{2,t-1}} - \lambda n_{2,t}\right)[1-\theta_t - (1-\omega)(1-\pi_t)]\,\frac{Y_t}{N_{2,t}}.$$

At the time of bargaining, the positive factors $\left(1 - \frac{\tau}{1+n_{i,t-1}} - \lambda n_{i,t}\right) Y_t/N_{i,t}$ are predetermined. Maximising over θ_t therefore gives

$$\theta_t = \gamma\omega + (1-\omega)\pi_t.$$

Assumption 1 sets $\omega = 0$. In that case no crop is lost in war, the feasible individually rational set collapses to the single allocation $\theta_t = \pi_t$, and both agreement surpluses are zero. Thus the usual positive-surplus Nash product is degenerate; $\theta_t = \pi_t = N_{1,t}/(N_{1,t} + N_{2,t})$ should be understood as the limit of the general solution as $\omega \downarrow 0$, not as the maximiser of an invalid zero-surplus Nash product.

4.4 Fertility

We can now compute fertility in the Assumption 1 special case. At time t, current population and the current sharing rule are predetermined. Group 1 therefore solves

$$\max_{n_{1,t}} \quad \underbrace{\left(1 - \frac{\tau}{1+n_{1,t-1}}\right)\frac{\theta_t Y_t}{N_{1,t}}}_{\text{constant at } t} + \frac{\beta\tau n_{1,t}}{1+n_{1,t}}\left[\frac{N_{1,t}n_{1,t}}{N_{1,t}n_{1,t} + N_{2,t}n_{2,t}}\right]\frac{A(R_{t+1})L}{N_{1,t}n_{1,t}}.$$

Group 2's current share is $1-\theta_t$, and its future share and future population are based on Group 2, not Group 1. Its corresponding problem is

$$\max_{n_{2,t}} \quad \underbrace{\left(1 - \frac{\tau}{1 + n_{2,t-1}}\right) \frac{(1-\theta_t)Y_t}{N_{2,t}}}_{\text{constant at } t} + \frac{\beta \tau n_{2,t}}{1 + n_{2,t}} \left[\frac{N_{2,t} n_{2,t}}{N_{1,t} n_{1,t} + N_{2,t} n_{2,t}} \right] \frac{A(R_{t+1})L}{N_{2,t} n_{2,t}}.$$

After cancelling future own-group population from each crop share and per-capita income, the fertility-dependent terms are proportional to

$$\frac{n_{i,t}}{(1 + n_{i,t})(N_{1,t} n_{1,t} + N_{2,t} n_{2,t})}.$$

The first-order conditions give the best responses

$$n_{1,t}^2 = \frac{N_{2,t}}{N_{1,t}} n_{2,t}, \qquad n_{2,t}^2 = \frac{N_{1,t}}{N_{2,t}} n_{1,t}.$$

Solving these two conditions simultaneously gives the equilibrium fertility levels

$$n_{1,t}^\star = \left(\frac{N_{2,t}}{N_{1,t}}\right)^{\frac{1}{3}},$$
$$n_{2,t}^\star = \left(\frac{N_{1,t}}{N_{2,t}}\right)^{\frac{1}{3}},$$

Thus Group 1's equilibrium fertility is higher when Group 2 is relatively more populous, and conversely. This strategic response is the population-race mechanism. In the paper's general model it can amplify total population and environmental pressure. Under Assumption 1, however, the population dynamics below equalise two finite clan populations; they do not generate population explosion. Population still affects the resource stock through

$$R_{t+1} = \left(1 + \delta - \delta \frac{R_t}{K} - b(N_{1,t} + N_{2,t})\right) R_t,$$

4.5 Steady-state population and natural resources

For the Assumption 1 special case, we can compute the steady-state population using the two dynamic equations:

$$n^{\star}_{1,t} = \left(\frac{N_{2,t}}{N_{1,t}}\right)^{\frac{1}{3}} \implies N_{1,t+1} = N_{1,t}\left(\frac{N_{2,t}}{N_{1,t}}\right)^{\frac{1}{3}}$$
$$n^{\star}_{2,t} = \left(\frac{N_{1,t}}{N_{2,t}}\right)^{\frac{1}{3}} \implies N_{2,t+1} = N_{2,t}\left(\frac{N_{1,t}}{N_{2,t}}\right)^{\frac{1}{3}},$$

It is simpler to solve the corresponding linearised system, which we can obtain by taking logarithms:

$$\tilde{N}_{i,t+1} = \frac{2}{3}\tilde{N}_{i,t} + \frac{1}{3}\tilde{N}_{j,t},$$

where $\tilde{N}_{i,t} = \log N_{i,t}$.

We can obtain the dynamics of the logarithmic system:

$$\tilde{N}_{i,t} = \frac{\tilde{N}_{i,0} + \tilde{N}_{j,0}}{2} + \frac{1}{2}3^{-t}\left(\tilde{N}_{i,0} - \tilde{N}_{j,0}\right).$$

The population of each clan converges to the finite common level

$$\bar{N}_i = \bar{N}_j = \sqrt{N_{1,0}}\sqrt{N_{2,0}},$$

If $b(\bar{N}_i + \bar{N}_j) < \delta$, the corresponding positive steady-state level of natural resources is

$$\bar{R} = K\left(1 - \frac{b(\bar{N}_i + \bar{N}_j)}{\delta}\right).$$

4.5.1 Simulated trajectory

Lastly, we can compute the trajectory of the special-case system for a set of parameters to visualise the evolution of the main variables. For instance, take $N_{1,0} = 9$, $N_{2,0} = 20$, $\delta = 0.08$, $K = 400$, $b = 0.0012$ and $R_0 = 300$.

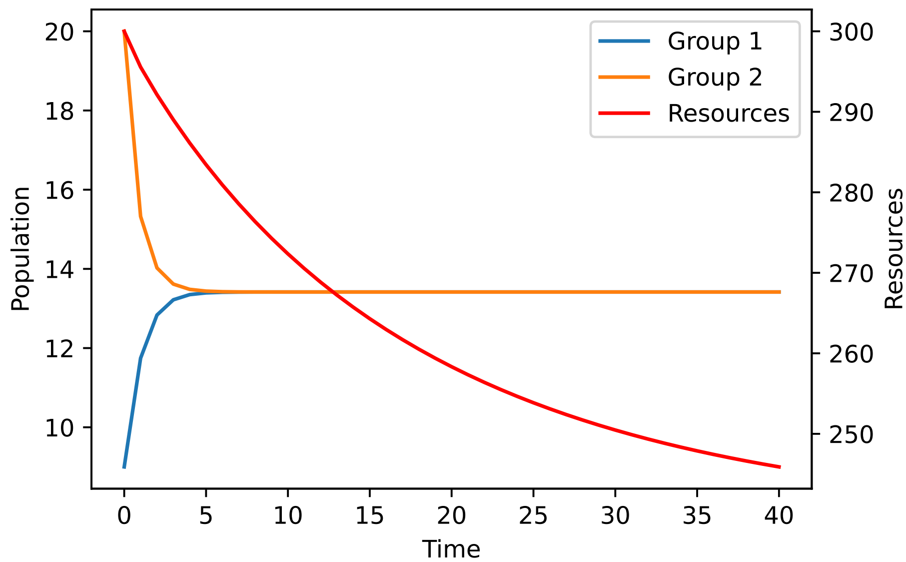

Figure 4.3: Simulated trajectory of populations and resources over time

5 Galor and Moav (2006)

This section discusses the work of Galor and Moav (2006), which presents an OLG model to explain the rise of publicly financed education and the transition from a class structure characterised by capitalists and workers to an economy in which both groups own capital.

The paper proposes that the demise of this class division was a deliberate action on the part of capitalists to sustain their profits as human capital became increasingly important in production. That is, at some point human capital becomes *really* necessary for production, and capitalists find it optimal to tax themselves to finance workers' education. Doing so raises the level of human capital and allows them to sustain their profits.

The mechanism is intended to describe the later stages of industrialisation and the transition to the beginning of the twentieth century. Figure 1 covers England from 1770 to 1920. It shows that inequality first rose during the early stages of industrialisation and then fell, while school enrolment increased strongly during the later decline in inequality. The theory the authors propose parallels this evolution. Furthermore, they supplement the model with econometrics that are compatible with the predictions of the model.

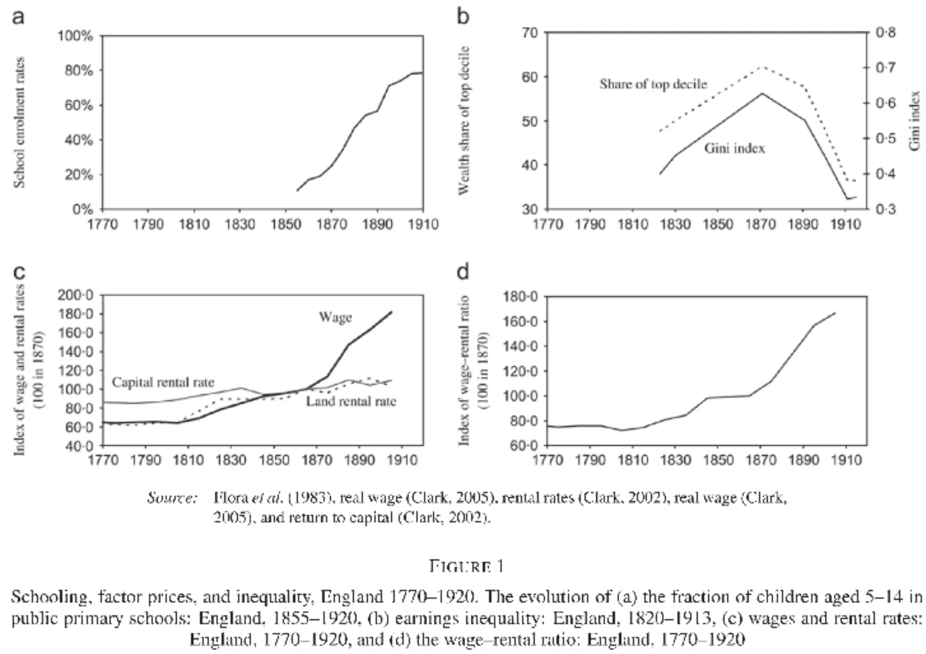

Figure 5.1: Figure 1 in Galor and Moav (2006)

5.1 The model

We present a simplified version of the model using precise functional forms for the human capital accumulation process and the production function. Our presentation comprises two sections: * The general model * The application of the model to a society with two classes. We will follow this approach.

For the moment, the economy comprises *only* one type of individual. Individuals live for two periods of time: young and adult. Young individuals do not produce and use their time to acquire human capital. If education is provided, human capital accumulation is faster.

5.1.1 Production and prices

A single homogeneous good is produced using physical and human capital according to a Cobb-Douglas production function. In particular:

$$Y_t = F(H_t, K_t) = AK_t^{\alpha} H_t^{1-\alpha} = H_t A k_t^{\alpha}, \quad k_t \equiv \frac{K_t}{H_t}.$$

Thus, aggregate output is $Y_t = H_t f(k_t)$, where output per efficiency unit is $f(k_t) = Ak_t^{\alpha}$.

Given the wage rate per efficiency unit of labour w_t and the return to capital r_t, producers maximise profits by choosing physical capital K_t and efficiency units of labour H_t; that is, $\{K_t, H_t\} = \arg\max \left[AH_t k_t^{\alpha} - w_t H_t - r_t K_t\right]$. Considering perfect competition, the inverse demand for each factor is:

$$r_t = f'(k_t) = \alpha A k_t^{\alpha-1} = r(k_t),$$
$$w_t = f(k_t) - f'(k_t)k_t = (1-\alpha)Ak_t^{\alpha} = w(k_t).$$

5.1.2 Individuals and preferences

Every period, a new generation of size 1 is born. Each individual has one parent and one child. Individuals live for two periods: during their youth, they accumulate human capital; and education improves human capital accumulation. Young individuals may receive a positive bequest from their parents on which they earn interest (the bequest is physical capital lent to producers). As adults, they supply their human capital as efficiency units of labour, receive interest on their assets, and allocate total income between consumption and a bequest.

The bequest b_t is transferred from parents to children, and the government collects a tax $\tau_t \geq 0$ on it. The remaining share $1 - \tau_t$ is invested as physical capital and generates income in adulthood. Physical capital **fully depreciates** between periods.

As mentioned, individuals accumulate human capital during their youth, and if they are provided with education (e_t), human capital accumulation is enhanced. However, even if no education is provided, all young individuals manage to obtain a minimum level of human capital: we set it equal to one. We model human capital accumulation as follows:[1]

$$h_{t+1} = 1 + \frac{e_t}{1 + e_t} = h(e_t).$$

This type of function guarantees a minimum level of human capital while ensuring that, under some conditions, investment in education is not optimal.

As adults, individuals receive wages on their human capital as well as the return on the untaxed part of their bequest. Therefore, an individual with a bequest b_t and education level e_t has income equal to:

$$I^i_{t+1} = w_{t+1}h(e_t) + (1 - \tau_t)b^i_t R_{t+1},$$

and, since capital fully depreciates, $R_{t+1} = r_{t+1} = r(k_{t+1})$.

Lastly, the preferences of adults include consumption and a taste for giving bequests. Bequests in the model are a type of luxury good: individuals

[1]Bisin and Verdier also model how parental indoctrination affects the evolution of cultural traits.

leave a positive bequest, $b_{t+1} > 0$, only when their income is sufficiently high. In particular:

$$u_t^i = (1-\beta)\log(c_{t+1}^i) + \beta\log(\bar{\theta} + b_{t+1}^i),$$

where $\bar{\theta} > 0$ and $\beta \in (0,1)$. The budget constraint is simple:

$$c_{t+1}^i + b_{t+1}^i = I_{t+1}^i.$$

5.1.3 Optimisation

We can easily compute the value of bequests, as a function of the income level. Replacing c_{t+1} in the objective function and taking the derivative with respect to b_{t+1} yields:

$$-\frac{1-\beta}{I_{t+1}^i - b_{t+1}^i} + \frac{\beta}{b_{t+1}^i + \bar{\theta}} = 0 \implies$$
$$b_{t+1}^i = \begin{cases} \beta(I_{t+1}^i - \theta) & \text{if} \quad I_{t+1}^i > \theta \\ 0 & \text{if} \quad I_{t+1}^i \leq \theta \end{cases}$$

where $\theta \equiv \bar{\theta}\frac{1-\beta}{\beta}$. Hence, when income is relatively low, individuals do not leave bequests.

5.1.4 Evolution of physical and human capital

Remember that bequests left during period t are the capital of period $t+1$. If B_t is the total amount of bequests left during t, then

$$K_{t+1} = (1-\tau)B_t.$$

The remaining τB_t goes to the government, which uses it to fund education. Population is normalised to 1, therefore, each individual receives education equal to: $e_t = \tau B_t$, and human capital evolves as:

$$H_{t+1} = h(e_t) = h(\tau_t B_t) = 1 + \frac{\tau_t B_t}{1 + \tau_t B_t}.$$

Finally, the level of $k_t = \frac{K_t}{H_t}$ is:

$$k_{t+1} = \frac{K_{t+1}}{H_{t+1}} = \frac{(1-\tau_t)B_t}{h(\tau_t B_t)} = \frac{(1-\tau_t)B_t}{1 + \frac{\tau_t B_t}{1+\tau_t B_t}} = k(\tau_t, B_t).$$

5.2 Optimal level of taxation

The paper assumes that the government selects a common proportional bequest tax. Voters compare common tax rates while internalising their effects on public education and competitive factor prices; aggregate bequests B_t and group membership are fixed when the rate is chosen, and education is provided equally to everyone. One important feature of the model is that utility is increasing in income I^i_{t+1}. We can easily check this by rewriting the indirect utility:

$$u^i_t = (1-\beta)\log(c^i_{t+1}) + \beta \log(\bar{\theta} + b^i_{t+1}) =$$
$$= \begin{cases} (1-\beta)\log\left((1-\beta)(I^i_{t+1} + \bar{\theta})\right) + \beta\log\left(\beta(I^i_{t+1} + \bar{\theta})\right) & \text{if} \quad I^i_{t+1} > \theta \\ (1-\beta)\log(I^i_{t+1}) + \beta\log(\bar{\theta}) & \text{if} \quad I^i_{t+1} \leq \theta \end{cases}$$

which is increasing in I^i_{t+1} because $\beta \in (0,1)$.

Therefore, instead of maximising the indirect utility, the government can maximise second-period income I^i_{t+1}, which in turn will maximise utility. Individual i's preferred common tax rate is denoted by τ^i_t. Second-period income under that rate is $w_{t+1}h(\tau^i_t B_t) + (1-\tau^i_t)b^i_t R_{t+1}$, where $w_{t+1} = w(k_{t+1})$ and $R_{t+1} = R(k_{t+1})$. At the same time, under individual i's preferred common rate, $k_{t+1} = \frac{(1-\tau^i_t)B_t}{h(\tau^i_t B_t)} = \frac{(1-\tau^i_t)B_t}{1+\frac{\tau^i_t B_t}{1+\tau^i_t B_t}}$. Putting everything together,

$$\begin{aligned}\tau_t^i &= \arg\max w_{t+1} h(\tau_t^i B_t) + (1-\tau_t^i) b_t^i R_{t+1} \\ &= \arg\max A(1-\tau_t^i)^\alpha h(\tau_t^i B_t)^{1-\alpha} B_t^\alpha \left(1-\alpha+\alpha\frac{b_t^i}{B_t}\right).\end{aligned}$$

Maximising with respect to τ_t^i yields:

$$\frac{\partial}{\partial \tau_t^i} = 0 \implies$$

$$AB_t^\alpha \left(1-\alpha+\alpha\frac{b_t^i}{B_t}\right)\left[-\alpha(1-\tau_t^i)^{\alpha-1} h(\tau_t^i B_t)^{1-\alpha} + \right.$$
$$\left. +(1-\alpha)h'(\tau_t^i B_t) h(\tau_t^i B_t)^{-\alpha} B_t (1-\tau_t^i)^\alpha\right] = 0$$

$$\alpha(1-\tau_t^i)^{\alpha-1} h(\tau_t^i B_t)^{1-\alpha} =$$
$$= (1-\alpha)h'(\tau_t^i B_t) B_t (1-\tau_t^i)^\alpha h(\tau_t^i B_t)^{-\alpha}$$

$$\alpha A(1-\tau_t^i)^{\alpha-1} h(\tau_t^i B_t)^{1-\alpha} B_t^{\alpha-1} =$$
$$= (1-\alpha)Ah'(\tau_t^i B_t) B_t^\alpha (1-\tau_t^i)^\alpha h(\tau_t^i B_t)^{-\alpha}$$

$$\underbrace{\alpha A(1-\tau_t^i)^{\alpha-1} h(\tau_t^i B_t)^{1-\alpha} B_t^{\alpha-1}}_{R(k_{t+1})} =$$
$$= \underbrace{(1-\alpha)AB_t^\alpha (1-\tau_t^i)^\alpha h(\tau_t^i B_t)^{-\alpha}}_{w(k_{t+1})} h'(\tau_t^i B_t)$$

$$R(k_{t+1}) = w(k_{t+1}) h'(\tau_t^i B_t)$$

The positive multiplier $AB_t^\alpha(1-\alpha+\alpha b_t^i/B_t)$ changes the level of income but not the tax rate that maximises it. Under the assumptions above, including identical preferences, universal education, competitive prices, and a common proportional tax, neither the interior condition nor the relevant boundary comparison depends on b_t^i. Consequently, all individuals prefer the same tax rate. This unanimity result need not hold if education or tax rates differ across individuals, or if voters do not internalise the policy's effect on factor prices. In our case, substituting and solving for τ_t:

$$\tau_t = \begin{cases} \frac{-B_t(1+2\alpha)+\sqrt{B_t^2(1+4(1+2B_t)(1-\alpha)\alpha)}}{4B_t^2\alpha} & \text{if} \quad B_t > \frac{\alpha}{1-\alpha} \\ 0 & \text{if} \quad B_t \leq \frac{\alpha}{1-\alpha} \end{cases} = \tau(B_t)$$

Alternatively, it is possible to re-express the condition for positive taxation in terms of k_{t+1} :

$$\tau_t = \begin{cases} \frac{-B_t(1+2\alpha)+\sqrt{B_t^2(1+4(1+2B_t)(1-\alpha)\alpha)}}{4B_t^2\alpha} & \text{if} \quad k_{t+1} > \frac{\alpha}{1-\alpha} \\ 0 & \text{if} \quad k_{t+1} \leq \frac{\alpha}{1-\alpha} \end{cases} = \tau(B_t)$$

5.3 One economy, two groups

We suppose now that the economy, at time $t = 0$, consists of two groups: capitalists (C) and workers (W). The capitalist share of the population is denoted by λ_t. Since every individual has one child, the group shares remain constant; that is, $\lambda_t = \lambda$. The only difference between the two groups is their initial endowment of capital: * Capitalists own the initial stock of capital (which we assume is large enough for them to leave bequests). * Workers have no capital and thus leave no bequests initially.

Therefore, the total amount of bequests in a given period is:

$$B_t = \lambda b_t^C + (1-\lambda)b_t^W.$$

The remainder of the model is the same as before, in particular,

$$k_{t+1} = \frac{(1 - \tau(B_t))B_t}{h(\tau(B_t)B_t)}.$$

Such an economy shifts from a two-class division based on capital ownership to one in which both groups own capital. The two population groups remain, and they need not hold equal wealth. The critical transition occurs because capitalists eventually find it optimal to impose a tax on themselves to finance public education. As a result, and as wages continue to increase, workers are eventually able to leave bequests and thus also become capital owners. Instead of detailing the exact process (see the reference), we will simulate the economy for a set of parameters.

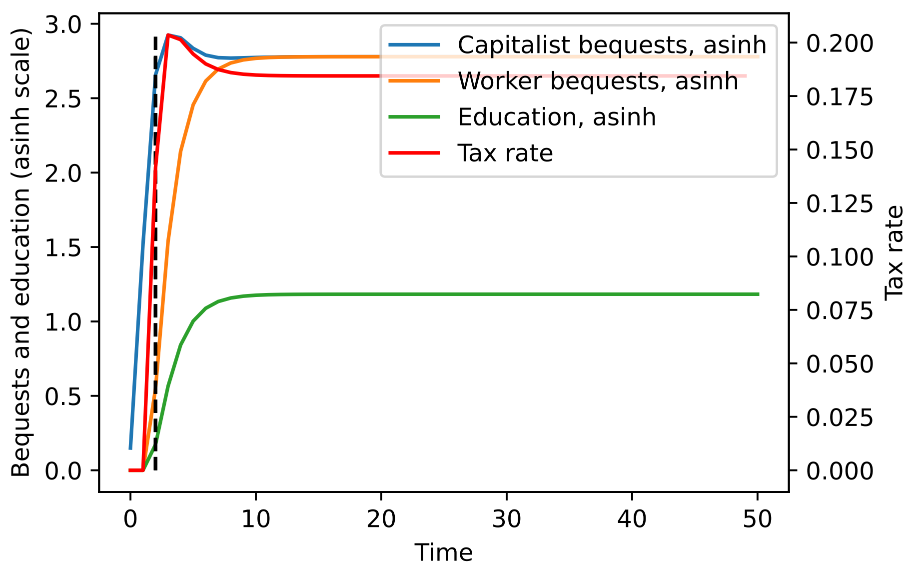

Figure 5.2: Simulated path of the economy proposed in Galor and Moav, 2006

6 Galor and Özak (2016)

Galor and Özak (2016) provide empirical evidence that differences in time preference (how much people discount the future) may have agricultural origins. This is important for development because more patient, future-oriented individuals generally have a greater propensity to save. Although the paper's major contribution is empirical, the authors develop an OLG model from which they derive testable implications.

The theoretical model shows how the composition of a population can be modelled using the OLG framework. The most influential model in that regard is Bisin and Verdier (2001). However, the model in Galor and Özak is relatively simple and illustrates well some population dynamics.

6.1 The model

We use an OLG framework and assume that the economy is agricultural and at a very early stage of development. In every period, the economy consists of individuals who live for **three** periods.

- During the first period of life, individuals are children and are economically passive; their parents provide their consumption.
- In the second and third periods of life, individuals work.
- All individuals can choose between two modes of production:
    - Endowment mode: it provides an equal payoff during the second and third periods of life. For instance, individuals may be hunters.

– Investment mode: it pays little during the second period of life, but the payoff during the third period is much larger. This represents farmers, who must sow seeds and wait for crops to grow.

Lastly, a crucial assumption of the model is the **lack of financial markets and long-term storage technology**. This implies that individuals **cannot** transfer consumption between periods two and three. Hence, output produced in each working period must be consumed in that period.

6.1.1 Production modes

All adults must choose either the *endowment mode* or the *investment mode*. The endowment mode provides a constant level of output, $R^0 > 1$, in each working period. If investment is instead chosen, it requires an investment during the first working period, which implies that fewer resources are available for consumption. In particular, we assume that it leaves individuals with 1 unit of consumption during their second period of life. However, the output it yields during the third period, R^1, is higher than under *endowment*: $\ln(R^1) > 2\ln(R^0)$.

Finally, depending on the chosen production mode, the income of individual i is given by:

$$(y_{i,t}, y_{i,t+1}) = \begin{cases} (R^0, R^0) & \text{if endowment} \\ (1, R^1) & \text{if investment} \end{cases}$$

6.2 Preferences

A key mechanism generating dynamics in the evolution of individual traits is the fertility decision. This approach is common: typically, fertility is linked to a trait through income. That is, individuals with more income will be able to have more children. If the trait is transmitted from parents to children, then the trait associated with greater income will become increasingly prevalent in the economy.[1] Preferences are therefore important in this type of model because fertility decisions are derived from them.

In every period t, a generation of size L_t becomes economically active; that is, it reaches the second period of life. Those individuals were born in period $t-1$. At this stage, each individual will live for two more periods. Recall that financial markets do not exist and that it is **impossible** to transfer resources between periods through storage.

We assume that, during the second period of life, individuals only consume what they produce. During the third and final period, individuals consume and have children. In particular, utility is given by:

$$u^{i,t} = \ln c_{i,t} + \beta_t^i[\gamma \ln n_{i,t+1} + (1-\gamma)\ln c_{i,t+1}], \quad \gamma \in (0,1),$$

where $c_{i,t}$ and $c_{i,t+1}$ are consumption in the second and third periods of life and $n_{i,t+1}$ is the number of children. The parameter $\beta_t^i \in (0,1]$ is individual i's discount factor: it measures how much the individual values the future relative to the present. The larger β_t^i is, the more the individual values the future and, hence, the more patient the individual is. Notice that β_t^i varies across individuals and can evolve over time.

During the second period, individuals do not really make any decision: since resources cannot be transferred, all production must be consumed. Hence, $c_{i,t} = y_{i,t}$. However, during the final period, individuals can trade off utility from consumption against utility from children. The paper assumes that each child costs τ units of consumption, which gives rise to the final-period budget constraint:

$$y_{i,t+1} = c_{i,t+1} + \tau n_{i,t+1}.$$

[1] Bisin and Verdier also model how parental indoctrination affects the evolution of cultural traits.

Given these preferences, utility maximisation implies:

$$c_{i,t+1} = (1-\gamma)y_{i,t+1},$$

$$n_{i,t+1} = \frac{\gamma}{\tau}y_{i,t+1}.$$

Lastly, the indirect utility ($v_{i,t}$) of individual i is given by:

$$v_{i,t} = \ln y_{i,t} + \beta_t^i[\ln y_{i,t+1} + \xi], \quad \xi \equiv \gamma \ln\left(\frac{\gamma}{\tau}\right) + (1-\gamma)\ln(1-\gamma).$$

6.3 Hunters or farmers

Since individuals can decide on their mode of production, they are free to choose to become either hunters or farmers. That is, each individual chooses the mode of production (endowment or investment) that maximises lifetime utility. Hence,

$$v_{i,t} = \begin{cases} \ln R^0 + \beta_t^i(\ln(R^0) + \xi) & \text{if endowment} \\ \ln 1 + \beta_t^i(\ln(R^1) + \xi) & \text{if investment} \end{cases}.$$

An individual is indifferent between modes of production if they obtain the same utility from both. That is, the individual with $\beta_t^i = \hat{\beta}$ is indifferent between becoming a hunter and becoming a farmer if and only if

$$\ln R^0 + \hat{\beta}(\ln(R^0) + \xi) = \ln 1 + \hat{\beta}(\ln(R^1) + \xi).$$

Solving for $\hat{\beta}$ allows us to identify this individual:[2]

$$\hat{\beta} = \frac{\ln R^0}{\ln R^1 - \ln R^0} \in (0,1).$$

[2]The assumption $\ln(R^1) > 2\ln(R^0)$ is important to establish that $\hat{\beta} \in (0,1)$.

Thus, all individuals with $\beta_t^i < \hat{\beta}$ optimally choose the endowment technology, while those with $\beta_t^i > \hat{\beta}$ find the investment technology optimal. Note that, as the return to agriculture increases (R^1 increases), the cutoff value $\hat{\beta}$ decreases:

$$\frac{\partial \hat{\beta}}{\partial R^1} = \frac{-\ln R^0}{R^1(\ln R^1 - \ln R^0)^2} < 0,$$

so, as agriculture becomes more profitable, more individuals find it optimal to become farmers.

Hence, we can rewrite an individual's income as a function of β_t^i:

$$(y_{i,t}, y_{i,t+1}) = \begin{cases} (R^0, R^0) & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} \leq \hat{} \\ (1, R^1) & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} > \hat{} \end{cases}.$$

Because final-period income differs, hunters and farmers have different numbers of children. In particular, using the optimal number of children derived above:

$$n_{i,t+1} = \frac{\gamma}{\tau} y_{i,t+1} = \begin{cases} \frac{\gamma}{\tau} R^0 \equiv n^E & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} \leq \hat{} \\ \frac{\gamma}{\tau} R^1 \equiv n^I & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} > \hat{} \end{cases}.$$

Because $R^1 > R^0$, farmers have more children than hunters.

6.4 The evolution of preferences

Finally, we can trace how preferences change over time due to the differential fertility of farmers and hunters. If time preferences β_t^i are transmitted from parents to children, farmers' higher fertility causes their share of the population to increase over time. The paper makes essentially this assumption but modifies the transmission of preferences for individuals engaged in farming. In particular: * β_t^i is perfectly transmitted if an individual is a hunter. * Farmers transmit a weakly larger value of β_t^i to their children, with equality at the fixed point, reflecting an acquired tolerance for waiting and delaying reward. Specifically,

$$\beta_{t+1}^i = \begin{cases} \beta_t^i & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} \leq \hat{} \\ \phi(\beta_t^i, R^1) & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} > \hat{} \end{cases},$$

where ϕ has the following properties:

- $\phi(\beta, R^1) < 1$ for $\beta \in [\hat{\beta}, 1]$, so the transmitted discount factor remains below one.
- Along the transition to the steady state, $\beta_t^i \leq \phi(\beta_t^i, R^1)$: the child's discount factor is weakly greater than the parent's. Equality means that the dynasty has reached a fixed point.
- $\phi(\hat{\beta}, R^1) > \hat{\beta}$, so transmission strictly raises the discount factor at the farming cutoff.
- $\phi_\beta(\beta, R^1) > 0$: the transmitted discount factor is increasing in the parent's discount factor.
- $\phi_{\beta\beta}(\beta, R^1) < 0$: the marginal effect of the parent's discount factor decreases as that factor rises.
- $\phi_R(\beta, R^1) > 0$: a higher agricultural return raises the discount factor transmitted to the child.

Suppose an individual at the beginning of time has $\beta_0^i < \hat{\beta}$. This individual optimally chooses to be a hunter, and the transmission process implies that the individual's children inherit $\beta_1^i = \beta_0^i$. Because $\hat{\beta}$ is constant over time, all descendants also choose to be hunters and transmit the same time preference. Hence,

$$\beta_0^i \leq \hat{\beta} \implies \lim_{t \to \infty} \beta_t^i = \beta_0^i.$$

Suppose instead that $\beta_0^i > \hat{\beta}$ and lies below the farmer-dynasty fixed point defined below. The individual becomes a farmer and transmits $\phi(\beta_0^i, R^1) > \beta_0^i$. All descendants are therefore also farmers, and their discount factors rise weakly until they converge to the fixed point:

$$\lim_{t \to \infty} \beta_t^i = \bar{\beta}^I.$$

Thus, $\bar{\beta}^I$ is the limiting discount factor for these farmer dynasties, not a maximum over every feasible initial discount factor.

6.4.1 Proof (not in the paper)

We want to show that $\beta_{t+1}^i = \phi(\beta_t^i, R^1)$ has a unique steady state on $[\hat{\beta}, 1]$. This amounts to showing that $\bar{\beta}^I = \phi(\bar{\beta}^I, R^1)$ for a unique $\bar{\beta}^I$ in this interval. Define

$$G(\beta) = \phi(\beta, R^1) - \beta.$$

The assumptions imply

$$G(\hat{\beta}) = \phi(\hat{\beta}, R^1) - \hat{\beta} > 0$$

and

$$G(1) = \phi(1, R^1) - 1 < 0.$$

Continuity therefore guarantees a root $\bar{\beta}^I \in (\hat{\beta}, 1)$. Moreover,

$$G''(\beta) = \phi_{\beta\beta}(\beta, R^1) < 0,$$

so G is strictly concave. Once a strictly concave function falls from the positive value $G(\hat{\beta})$ to a root, its subsequent secant slopes are negative; it therefore remains below zero and cannot cross the horizontal axis again. The root $\bar{\beta}^I$ is thus unique. For $\beta_0^i \in (\hat{\beta}, \bar{\beta}^I]$, the transmission sequence is nondecreasing and, because ϕ is increasing, bounded above by $\bar{\beta}^I$. Its limit must satisfy the fixed-point equation and hence equals $\bar{\beta}^I$.

6.5 Evolution of traits over time

Lastly, suppose that, at time $t = 0$, the population has different levels of time preference. We assume that initial traits are characterised by a density $\eta(\beta_0^i)$ with support $[0, \bar{\beta}^I]$. Furthermore, we normalise the initial generation to be of size one: $L_0 = 1$. Equivalently,

$$L_0 = \int_0^{\bar{\beta}^I} \eta(\beta_0^i)\, \mathrm{d}\beta_0^i = 1.$$

We also know that all individuals whose $\beta_0^i \leq \hat{\beta}$ decide to use the endowment technology, while the remaining individuals opt for the investment technology. Therefore, the sizes of the hunter (E) and farmer (I) groups are given by

$$L_0^E = \int_0^{\hat{\beta}} \eta(\beta_0^i)\, \mathrm{d}\beta_0^i,$$

$$L_0^I = \int_{\hat{\beta}}^{\bar{\beta}^I} \eta(\beta_0^i)\, \mathrm{d}\beta_0^i.$$

The number of individuals evolves according to each group's fertility rate:

$$L_t^E = L_0^E (n^E)^t = \left(\frac{\gamma}{\tau} R^0\right)^t L_0^E,$$

$$L_t^I = L_0^I (n^I)^t = \left(\frac{\gamma}{\tau} R^1\right)^t L_0^I,$$

and total population is $L_t = L_t^E + L_t^I$.

Finally, notice that the distribution of β_t^i does not change within the endowment group, whose members have $\beta_t^i \leq \hat{\beta}$: they all have the same number of children, and each child inherits the parent's trait. Therefore, the average value $\bar{\beta}_t^E$ is constant over time. In contrast, the average value $\bar{\beta}_t^I$ increases toward $\bar{\beta}^I$. At any period t, the overall average value of the discount factor is given by

$$\bar{\beta}_t = \theta_t^E \bar{\beta}_t^E + (1 - \theta_t^E) \bar{\beta}_t^I,$$

where θ_t^E is the fraction of individuals who engage in the endowment production process and $\bar{\beta}_t$ is the population-weighted average discount factor.

$$\theta_t^E = \frac{L_t^E}{L_t^E + L_t^I} = \frac{(R^0)^t}{(R^0)^t + (R^1)^t \frac{L_0^I}{L_0^E}}.$$

Hence, as time advances, the share of the population engaged in the endowment production process approaches zero:

$$\lim_{t \to \infty} \theta_t^E = 0.$$

This process reflects the endowment group's lower reproductive success.

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